



Problem of the Week

Problem E and Solution

Llama Mall

**Problem**

The title, “Llama Mall”, is an example of a palindrome, a phrase that is the same when read forwards or backwards. Single words like MOM and BOB are palindromes. Numbers like 7, 414 and 12321 are also palindromes. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?

Solution

Our first observation is that since we are adding two four-digit palindromes to form a five-digit palindrome, then for some digits a, b, c, d, e, f, g , we must have

$$\begin{array}{r} a b b a \\ + c d d c \\ \hline e f g f e \end{array}$$

Also, since this five-digit number is the sum of two four-digit numbers, e must be 1. So we now have,

$$\begin{array}{r} a b b a \\ + c d d c \\ \hline 1 f g f 1 \end{array}$$

From the units column we know that $a + c$ has a units digit of 1. Also, since a and c are digits, then $a + c < 20$. It cannot be the case that $a + c = 1$, because then otherwise either $a = 0$ or $c = 0$ (and then we are not adding two four-digit palindromes). Thus, $a + c = 11$. We list the possibilities for a and c :

a	2	3	4	5
c	9	8	7	6

Note that there are only four possibilities here. If we extended this table, we would get an additional four possibilities which would be duplicates of these four, with a and c reversed.



Since $a + c = 11$, the first two digits of the sum are either 11 (if there is no carry from the hundreds column) or 12 (if there is a carry from the hundreds column). This means either $f = 1$ or $f = 2$. Let's look at both options.

Option 1: $f = 1$

$$\begin{array}{r} a b b a \\ + c d d c \\ \hline 1 1 g 1 1 \end{array}$$

Since $f = 1$, then there is no carry from the hundreds column. Since there is no carry, this means that $b + d$ is a single digit, and from the tens digit of the sum we get $b + d + 1 = 1$. Thus $b + d = 0$, so $b = 0$ and $d = 0$. Notice that for each of the four possibilities for a and c , selecting $b = d = 0$ will generate a valid pair of four-digit palindromes. Therefore, there are a total of 4 four-digit palindrome pairs where $f = 1$.

Option 2: $f = 2$

$$\begin{array}{r} a b b a \\ + c d d c \\ \hline 1 2 g 2 1 \end{array}$$

Since $f = 2$, then there is a carry from the hundreds column. From the tens column, $b + d + 1$ must end in 2. Also, since there is a carry from the hundreds column, then $b + d + 1 = 12$, so $b + d = 11$. There are eight ways for this to happen:

b	2	3	4	5	6	7	8	9
d	9	8	7	6	5	4	3	2

Notice that provided $a + c = 11$, each of these 8 possibilities for b and d will produce a valid pair of four-digit palindromes. Therefore, for each of the four possibilities for a and c , there are 8 ways to produce a valid four-digit palindrome pair. Therefore, there are a total of $4 \times 8 = 32$ four-digit palindrome pairs where $f = 2$.

Therefore, the total number of pairs of four-digit palindromes that sum to a five-digit palindrome is $4 + 32 = 36$.

