



Problem of the Week

Problem E and Solution

Four Tokens Please

Problem

In a game, players compete to earn points. The banker “pays” the players using tokens. A circular token is worth 1 point, a square token is worth 5 points, a triangular token is worth 10 points, and a hexagonal token is worth 25 points. When paying a player, the banker must follow the minimum token rule. That is, the banker must pay using the least number of tokens possible. For example, to pay a player 30 points, there are many combinations of tokens that work. The banker could use 30 circular tokens (30×1 point = 30 points) or 2 square tokens and 2 triangular tokens (2×5 points + 2×10 points = 30 points), a total of four tokens. There are other possible ways to combine tokens to obtain 30 points. However, since the banker must pay using the minimum token rule, the banker would pay using 2 tokens, 1 hexagonal token and 1 square token (1×25 points + 1×5 points = 30 points). How many different point totals can the banker generate using exactly four tokens, provided the banker follows the minimum token rule?

Solution

For the solution, we will consider cases to carefully count the possibilities.

4 circular tokens

There is only one possibility here: four circular tokens gives a total of 4 points.

3 circular tokens

There are three possibilities here: three circular tokens and one square token giving a total of 8 points, three circular tokens and one triangular token giving a total of 13 points, and three circular tokens and one hexagonal token giving a total of 28 points.

2 circular tokens

- and two of one other token.

This leads to two possibilities: two circular tokens and two triangular tokens for a total of 22 points, and two circular tokens and two hexagonal tokens for a total of 52 points. Note that two circular tokens and two square tokens gives a total of 12 points. This same total can be made using fewer tokens, namely two circular tokens and one triangular token, and is therefore not valid.

- and one of each of two other tokens.

This leads to three valid possibilities: two circular tokens, one square token and one triangular for a total of 17 points; two circular tokens, one square token and one hexagonal token for a total of 32 points; and two circular tokens, one triangular token and one hexagonal token for a total of 37 points.

1 circular token

- and three of one other type of token.

This leads to one valid possibility: one circular token and three hexagonal tokens for a sum of 76 points. The possibilities one circular token and three square tokens, and one circular token and three triangular tokens are both invalid since the totals produced in each case can be made using fewer tokens. (This is left for the solver to verify.)





1 circular token (continued)

- and two of a second type of token and one of a third type of token.
This leads to three valid possibilities: one circular token, one square token and two hexagonal tokens for a total of 56 points; one circular token, one triangular token and two hexagonal tokens for a total of 61 points; and one circular token, one hexagonal token and two triangular tokens for a total of 46 points. The totals obtained using one circular token, one square token and two triangular tokens, or one circular token, one triangular token and two square tokens, or one circular token, one hexagonal token and two square tokens can be obtained using fewer tokens and hence are invalid. (Again, this is left for the solver to verify.)
- and one of each of the other three tokens.
This leads to one valid possibility: one circular token, one square token, one triangular token and one hexagonal token for a total of 41 points.

No circular tokens

- and four of one other token.
There is only one valid possibility here: using four hexagonal tokens, we obtain a total of 100 points. The other two possibilities made using 4 of the same token are both invalid since the totals can be made using fewer tokens. Four square tokens gives a total of 20 points. The same total can be made using two triangular tokens and hence fewer tokens. Four triangular tokens gives a total of 40 points. The same total can be made using one hexagonal token, one triangular token and one square token, and hence fewer tokens.
- with three of a second type of token and one of a third type of token.
This leads to two valid possibilities: three hexagonal tokens and one square token for a total of 80 points, and three hexagonal tokens and one triangular token for a total of 85 points. Using three square tokens and any other token would be invalid since three square tokens could be replaced with two tokens, a triangular token and a square token, to produce the same total points. Using three triangular tokens and any other token would be invalid since three triangular tokens could be replaced with two tokens, a hexagonal token and a square token, to produce the same total points.
- with two of one type of token and two of another type of token.
This leads to one valid possibility: two triangular tokens and two hexagonal tokens for a total of 70 points. Two square tokens could never be paired with any other pair of tokens since two square tokens can be replaced by one triangular token, and hence fewer tokens to produce the same total.
- two of one type of token and one of each of the remaining two different tokens.
This leads to one valid possibility: two hexagonal tokens, one square token, and one triangular token for a total of 65 points. Two triangular tokens, one square token and one hexagonal token gives a total of 50 points. This total can be made with fewer tokens by using two hexagonal tokens. Two square tokens, one triangular token and one hexagonal token gives a total of 45 points. This total can be made with fewer tokens by using one hexagonal token and two triangular tokens.

The number of different totals is the obtained by adding the number of possibilities from each case, namely $1 + 3 + 2 + 3 + 1 + 3 + 1 + 1 + 2 + 1 + 1 = 19$ possible totals.

The 18 possible totals using exactly four tokens are

4, 8, 13, 17, 22, 28, 32, 37, 41, 46, 52, 56, 61, 65, 70, 76, 80, 85, 100.

