Problem of the Week

Problem E and Solution

No Sevens

Problem

Determine the sum of all integers between 1 and 2019, inclusive, that do not contain the digit 7.

Solution

Solution 1

In this solution, we will use the fact that the sum of the integers from 1 to n is $\frac{n(n+1)}{2}$.
Consider first the integers from 1 to 100. The sum of these integers is $\frac{(100)(101)}{2} = 5050$.
The integers from 1 to 100 which do contain the digit 7 are 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97, whose sum is 1188. Therefore, the sum of the integers from 1 to 100 which do not contain the digit 7 is $5050 - 1188 = 3862$. There are 81 integers from 101 to 200 not containing the digit 7 as well. Each of these is 100 more than a corresponding integer between 1 and 100 which does not contain the digit 7, so the sum of these 81 integers is $3862 + 81(100)$.
We can use this approach to determine the sum of the appropriate numbers in each range of 100, as shown in the table:

Range	Number of Integers in Range that do not contain a 7	Sum of Integers in Range that do not contain a 7
1 to 100	81	3862
101 to 200	81	$3862 + 81(100)$
201 to 300	81	$3862 + 81(200)$
301 to 400	81	$3862 + 81(300)$
401 to 500	81	$3862 + 81(400)$
501 to 600	81	$3862 + 81(500)$
601 to 700	81	$3862 + 81(600) - 700$
701 to 800	1	800
801 to 900	81	$3862 + 81(800)$
901 to 1000	81	$3862 + 81(900)$
1001 to 1100	81	$3862 + 81(1000)$
1101 to 1200	81	$3862 + 81(1100)$
1201 to 1300	81	$3862 + 81(1200)$
1301 to 1400	81	$3862 + 81(1300)$
1401 to 1500	81	$3862 + 81(1400)$
1501 to 1600	81	$3862 + 81(1500)$
1601 to 1700	81	$3862 + 81(1600) - 1700$
1701 to 1800	1	1800
1801 to 1900	81	$3862 + 81(1800)$
1901 to 2000	81	$3862 + 81(1900)$

For 2001 to 2019, the only integers that contain the digit 7 are 2007 and 2017. Thus, the sum of the integers from 2001 to 2019 that do not contain a 7 is

$$19 \times 2000 + \frac{(19)(20)}{2} - 2007 - 2017 = 34\,166.$$

Therefore, the overall sum of the integers from 1 to 2019 that do not contain the digit 7 is

$$18(3862) + 81(16\,600) - 700 + 800 - 1700 + 1800 + 34\,166 = 1\,448\,482.$$

Solution 2

Consider first the integers from 000 to 999 that do not contain the digit 7. (We can include 000 in this list as it will not affect the sum.)
Since each of the three digits has 9 possible values, there are $9 \times 9 \times 9 = 729$ such integers.
If we fix any specific digit in any of the three positions, there will be exactly 81 integers with that digit in that position, as there are 9 possibilities for each of the remaining digits. (For example, there are 81 such integers ending in 0, 81 ending in 1, etc.)
We sum these integers by first summing the units digits, then summing the tens digits, and then summing the hundreds digits.
Since each of the 9 possible units digits occurs 81 times, the sum of the units digits column is

$$81(0) + 81(1) + 81(2) + 81(3) + 81(4) + 81(5) + 81(6) + 81(8) + 81(9) = 81(38).$$

Since each of the 9 possible tens digits occurs 81 times, the sum of the tens digits column is

$$81(0 + 10 + 20 + 30 + 40 + 50 + 60 + 80 + 90) = 81(380).$$

Similarly, the sum of the hundreds digits column is 81(3800).
Thus, the sum of the integers from 0 to 999 that do not contain the digit 7 is

$$81(38) + 81(380) + 81(3800) = 81(38)(1 + 10 + 100) = 81(38)(111) = 341\,658.$$

Each of the 729 integers from 1000 to 1999 which do not contain 7 is 1000 more than such an integer between 0 and 999. (There are again 729 of these integers as the first digit is fixed at 1, and each of the remaining three digits has 9 possible values.) Thus, the sum of these integers from 1000 to 1999 is equal to the sum of the corresponding ones from 0 to 999 plus 729(1000), or

$$341\,658 + 729\,000 = 1\,070\,658.$$

For 2000 to 2019, the only integers that contain the digit 7 are 2007 and 2017. Thus, the sum of the integers from 2000 to 2019 that do not contain the digit 7 is

$$20 \times 2000 + \frac{(19)(20)}{2} - 2007 - 2017 = 36\,166.$$

Therefore, the sum of the integers from 1 to 2019 that do not include the digit 7 is

$$341\,658 + 1\,070\,658 + 36\,166 = 1\,448\,482.$$