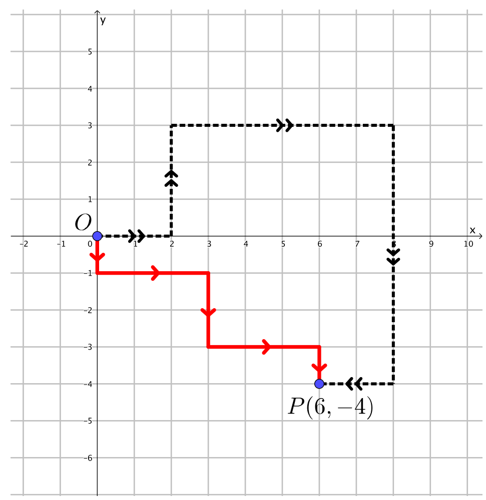
Problem of the Week
Problem E and Solution
Getting There

Problem

We define the *path length* between two points A and B as the minimum length along the grid lines from A to B. In the diagram, O is the origin and P is the point $(6, -4)$. The path length from O to P is 10. How many points with integer coordinates have a path length of 10 from O?

Solution
Solution 1

Let $Q(a, b)$ be a point that has path length of 10 from O, the origin.



Let's first assume that Q is on the x or y axis.
The only point along the positive x-axis that has path length 10 from the origin is $(10, 0)$.
The only point along the negative x-axis that has path length 10 from the origin is $(-10, 0)$.
The only point along the positive y-axis that has path length 10 from the origin is $(0, 10)$.
The only point along the negative y-axis that has path length 10 from the origin is $(0, -10)$.
Therefore, there are 4 points along the axes that have a path length of 10 from O.

Next, let's assume $a > 0$ and $b > 0$, so Q is in the first quadrant.
Since the path length from O to Q is 10, there must be a path from O to Q that moves a total of r units to the right and u units up (in some order) such that $r + u = 10$. This means that Q is r units to the right of O and u units up from O. In other words, $a = r$ and $b = u$, so $a + b = r + u = 10$.

The points (a, b) in the first quadrant that satisfy $a + b = 10$ where a and b are integers are $(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)$. There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path length of 10 from O.

By symmetry, there are 9 points in each of the four quadrants that have path length of 10 from O.
In quadrant 2, the points are

$$(-1, 9), (-2, 8), (-3, 7), (-4, 6), (-5, 5), (-6, 4), (-7, 3), (-8, 2), (-9, 1).$$

In quadrant 3, the points are

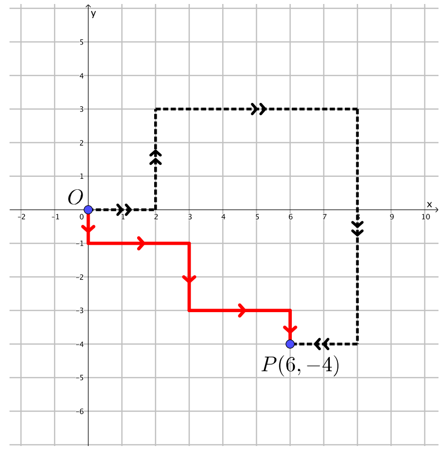
$$(-1, -9), (-2, -8), (-3, -7), (-4, -6), (-5, -5), (-6, -4), (-7, -3), (-8, -2), (-9, -1).$$

In quadrant 4, the points are

$$(1, -9), (2, -8), (3, -7), (4, -6), (5, -5), (6, -4), (7, -3), (8, -2), (9, -1).$$

Therefore, there are a total of $4 + (4 \times 9) = 40$ points with integer coordinates that have a path length of 10 from O.

Solution 2



We are permitted ten moves to get from the origin to a point by traveling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We will examine the cases by picking the number of horizontal moves. Then we will determine the corresponding number of vertical moves and the resulting possible endpoints.

- **0 horizontal moves**: Since there are no horizontal moves, there would be 10 vertical moves. There are two possible endpoints, $(0, 10)$, and $(0, -10)$.
- **1 horizontal move**: Since there is 1 horizontal move, there would be 9 vertical moves. There are four possible endpoints, $(-1, 9), (-1, -9), (1, 9)$, and $(1, -9)$.
- **2 horizontal moves**: Since there are 2 horizontal moves, there would be 8 vertical moves. There are four possible endpoints, $(-2, 8), (-2, -8), (2, 8)$, and $(2, -8)$.
- **3 horizontal moves**: Since there are 3 horizontal moves, there would be 7 vertical moves. There are four possible endpoints, $(-3, 7), (-3, -7), (3, 7)$, and $(3, -7)$.
- **4 horizontal moves**: Since there are 4 horizontal moves, there would be 6 vertical moves. There are four possible endpoints, $(-4, 6), (-4, -6), (4, 6)$, and $(4, -6)$.
- **5 horizontal moves**: Since there are 5 horizontal moves, there would be 5 vertical moves. There are four possible endpoints, $(-5, 5), (-5, -5), (5, 5)$, and $(5, -5)$.
- **6 horizontal moves**: Since there are 6 horizontal moves, there would be 4 vertical moves. There are four possible endpoints, $(-6, 4), (-6, -4), (6, 4)$, and $(6, -4)$.
- **7 horizontal moves**: Since there are 7 horizontal moves, there would be 3 vertical moves. There are four possible endpoints, $(-7, 3)$, $(-7, -3)$, $(7, 3)$, and $(7, -3)$.
- **8 horizontal moves**: Since there are 8 horizontal moves, there would be 2 vertical moves. There are four possible endpoints, $(-8, 2)$, $(-8, -2)$, $(8, 2)$, and $(8, -2)$.
- **9 horizontal moves**: Since there are 9 horizontal moves, there would be 1 vertical move. There are four possible endpoints, $(-9, 1)$, $(-9, -1)$, $(9, 1)$, and $(9, -1)$.
- **10 horizontal moves**: Since there are 10 horizontal moves, there would be 0 vertical moves. There are two possible endpoints, $(-10, 0)$, and $(10, 0)$.

Therefore, there are a total of $2 + (4 \times 9) + 2 = 40$ points with integer coordinates that have a path length of 10 from O.