Problem of the Week
Problem E and Solution
A Square Formation

Problem

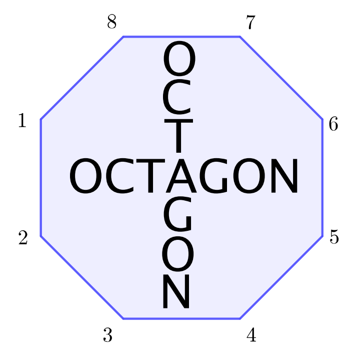
The vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and H and each letter is used exactly once, What is the probability that $ABCD$ is a square?

Solution

Solution 1

In order to determine the probability, we need to determine the number of ways to label the vertices of the regular octagon so that $ABCD$ forms a square and divide by the total number of ways the regular octagon can be labelled.

First, let's determine the total number of ways that the vertices of a regular octagon can be labelled A, B, C, D, E, F, G, H, in some order.



Let's start with the vertex on the top left. There are 8 possible ways to label it (it can be labelled as A, B, C, D, E, F, G or H). Moving clockwise, the next vertex can be labelled 7 different ways (it can be assigned any letter other than the letter assigned to the previous vertex). Moving clockwise, the next vertex can be labelled 6 different ways (it can be assigned any letter other than the 2 that have already been used). Moving clockwise, the next vertex can be assigned 5 different letters, and so on. Once we reach the last vertex there will only be 1 letter left, so it can be assigned a letter only 1 way.

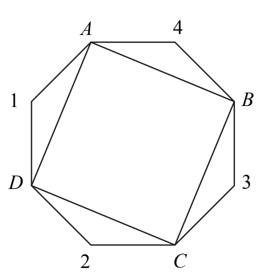
Therefore, there are

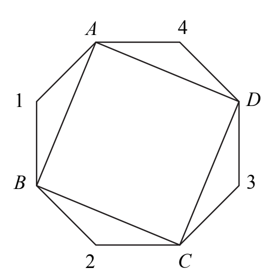
$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40\,320$$

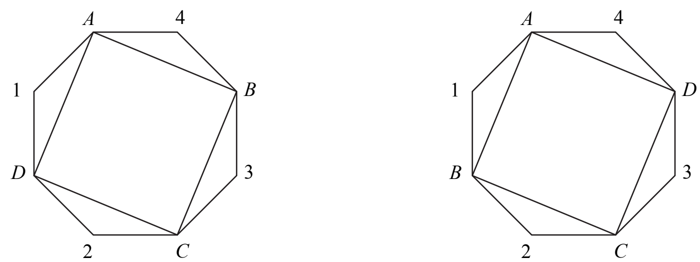
different ways to label the regular octagon with the letters $A, B, C, D, E, F, G,$ and H in some order.

Now, let's determine how many of the $40\,320$ labellings result in $ABCD$ forming a square.

Let's suppose vertex A is on the top left corner. Then there are two possible ways to label B, C and D so that $ABCD$ forms a square. They are shown below.







For each of these two cases, how many ways can the remaining 4 vertices be labelled? There are 4 choices for labelling the vertex to the right of A (it can be assigned E, F, G or H). Given the labelling of that vertex, moving clockwise, there are 3 choices for the next vertex, then 2 choices for the next and 1 choice for the last vertex.

Therefore, for each of the cases above, there are $4 \times 3 \times 2 \times 1 = 24$ ways to label the remaining vertices. Therefore, there are $24 + 24 = 48$ ways to label the regular octagon with A in the top left corner and $ABCD$ forming a square.

Using a similar argument, we can see that for any vertex that A can be assigned to, there will be 48 ways to label the regular octagon so that $ABCD$ forms a square. Since A can be assigned to 8 different vertices, there are $8 \times 48 = 384$ different ways to label the regular octagon so that $ABCD$ forms a square.

Therefore, the probability that $ABCD$ forms a square is $\frac{384}{40\,320} = \frac{1}{105}$.

Solution 2

The first solution counts the number of ways to create square $ABCD$ and divides by the total number of possible arrangements. This solution uses a more direct probability argument. Since the square labels A, B, C, and D are sequential, the A can go anywhere.

There is now a $\frac{2}{7}$ chance that the B will be placed in a location to create a square (either clockwise or counterclockwise around), given where the A is.

There is now a $\frac{1}{6}$ chance that the C will be placed in the only valid location, across from the A, and given that B is in an acceptable location.

There is now a $\frac{1}{5}$ chance that the D will be placed in the only valid location, given the $A, B,$ and C are located appropriately.

The remaining assignments are irrelevant to square $ABCD$.

By multiplying the probabilities, we obtain the probability that $ABCD$ forms a square is

$$\frac{2}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{2}{210} = \frac{1}{105}.$$