2019

Problem of the Week
Problem E and Solution
Triple Number Sums

Problem

The set $\{3, 6, 9, 12, 15, \cdots, 2016, 2019\}$ contains all of the multiples of three from 3 to 2019. Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?

Solution

Since the set includes every positive multiple of three from 3 to 2019 and 2019 is the largest number, then there are $2019 \div 3 = 673$ numbers in the set. Each number is of the form $3n$, for $n = 1, 2, 3, \cdots, 673$. The required sum is $3a + 3b + 3c$ where a, b, and c are three distinct numbers chosen from $\{1, 2, 3, \cdots, 673\}$. But $3a + 3b + 3c = 3(a + b + c)$. We can reduce the problem to the much easier question of, "How many distinct integers can be formed by adding three numbers from $\{1, 2, 3, \cdots, 673\}$?"

Clearly, the smallest number is $1 + 2 + 3 = 6$ and the largest number is $671 + 672 + 673 = 2016$. It is reasonably easy to see that it is possible to get every number in between 6 and 2016 by:

a) increasing the sum by replacing a number with one that is 1 larger or,
b) decreasing the sum by replacing a number with one that is 1 smaller.

Therefore, all of the numbers from 6 to 2016 inclusive can be formed. The number of numbers that can be formed is 2011. (Some solvers may think that there are 2010 numbers. There are 2016 integers from 1 to 2016, inclusive. But this includes the five numbers 1 to 5. So there are $2016 - 5 = 2011$ numbers from 6 to 2016.)

This answer, 2011, is the answer to the original problem as well. If $a + b + c = 6$ then $3(a + b + c) = 18$. This is the smallest number that is the sum of the three smallest numbers, 3, 6 and 9, from the original set. If $a + b + c = 2016$ then $3(a + b + c) = 6048$. This is the largest number that is the sum of the three largest numbers, 2013, 2016, and 2019, from the original set. Then every multiple of three from 18 to 6048 can be generated by adding three different numbers from the original set. (There are 2011 multiples of three from 18 to 6048, inclusive. And each of these can be obtained by adding three distinct numbers from the original set.)