Problem of the Week

Problem E and Solution

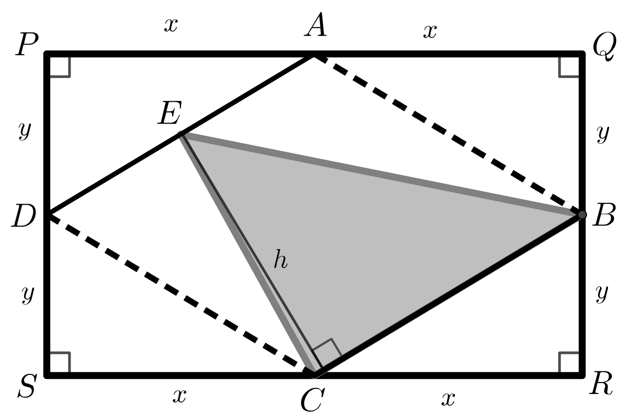
Always True?

Problem

$PQRS$ is a rectangle with $PQ = SR$ and $PS = QR$. The points A, B, C, and D are the midpoints of sides PQ, QR, RS, and SP, respectively. The point E is the midpoint of line segment AD. Show that it is always true that the area of rectangle $PQRS$ is four times the area of $\triangle BCE$.

Solution

Solution 1



Let the length of PQ be $2x$ and the length of PS be $2y$. It follows that

$$PA = AQ = SC = CR = x$$
$$\text{and } PD = DS = QB = BR = y$$

The area of rectangle

$$PQRS = PQ \times PS = (2x)(2y) = 4xy.$$

Therefore, we need to show that the area of $\triangle BCE = \frac{1}{4}(4xy) = xy$.

Since $PQRS$ is a rectangle, each of the corner angles is 90° and each of the four corner triangles are right triangles. Also, each of the four corner triangles, $\triangle APD$, $\triangle BQA$, $\triangle CRB$, and $\triangle DSC$, has base x and height y so their areas are the same. The total area of these four triangles is $4 \times \left(\frac{xy}{2}\right) = 2xy$.

The length of the hypotenuse of each of the four right triangles is $\sqrt{x^2 + y^2}$, so $AB = BC = CD = DA = \sqrt{x^2 + y^2}$ and $ABCD$ is a rhombus. Then, it follows that $AD \| CB$.

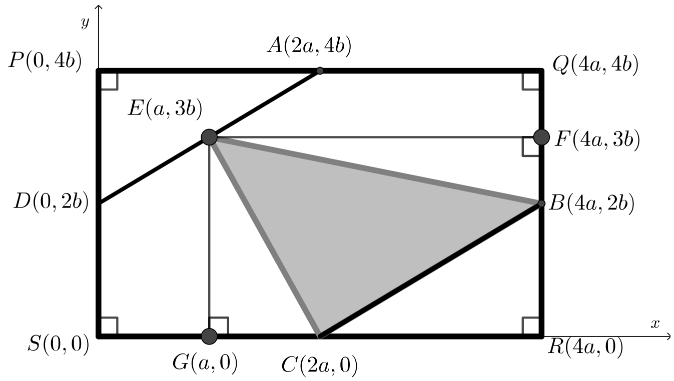
The area of rhombus $ABCD$ is equal to the area of $PQRS$ minus the area of the four corner right triangles. So, the area of rhombus $ABCD = 4xy - 2xy = 2xy$.

Let h be the perpendicular distance between AD and CB, two opposite parallel sides of the rhombus. The area of rhombus $ABCD = h \times BC$, but the area of rhombus $ABCD = 2xy$, so $h \times BC = 2xy$.

Then the area of $\triangle BCE = \frac{h \times BC}{2} = \frac{2xy}{2} = xy$.
Now, the area of $PQRS = 4xy = 4 \times$ the area of $\triangle BCE$, as required.

Solution 2

In this solution we will use analytic geometry.



Let the length of PQ be $4a$ and the length of PS be $4b$. It follows that

$$PA = AQ = SC = CR = 2a \text{ and } PD = DS = QB = BR = 2b.$$

Position S at the origin, R along the positive x-axis at $(4a, 0)$, and P along the positive y-axis at $(0, 4b)$. It then follows that Q is at $(4a, 4b)$. The midpoints are then $A(2a, 4b)$, $B(4a, 2b)$, $C(2a, 0)$, and $D(0, 2b)$. Since E is the midpoint of AD it is located at $(a, 3b)$.

Construct a line segment from E perpendicular to QR, intersecting QR at $F(4a, 3b)$. Construct a line segment from E perpendicular to SR, intersecting SR at $G(a, 0)$. It is easily shown that $EFRG$ is a rectangle.

$$\begin{aligned}
\text{Area } \triangle BCE &= \text{Area } EFRG - \text{Area } \triangle EFB - \text{Area } \triangle BRC - \text{Area } \triangle EGC \\
&= GR \times EG - \frac{EF \times FB}{2} - \frac{BR \times CR}{2} - \frac{EG \times GC}{2} \\
&= (3a)(3b) - \frac{(3a)(b)}{2} - \frac{(2b)(2a)}{2} - \frac{(3b)(a)}{2} \\
&= \frac{18ab - 3ab - 4ab - 3ab}{2} \\
&= 4ab \\
\text{Area } PQRS &= PQ \times PS \\
&= (4a)(4b) \\
&= 16ab \\
&= 4(4ab) \\
&= 4 \times \text{the area of } \triangle BCE, \text{ as required.}
\end{aligned}$$