

Problem of the Week

Problem E and Solution

Circle, Circle, Circle

Problem

AB is a diameter of a circle centred at O . A line segment is drawn from a point C on the circumference of the circle to D on AB such that $CD \perp AB$ and $CD = \sqrt{3}$ units. Two circles are drawn on AB . One has diameter AD and the other has diameter DB . Determine the area of the shaded region. That is, determine the area outside of the two inner circles but inside the outer circle.

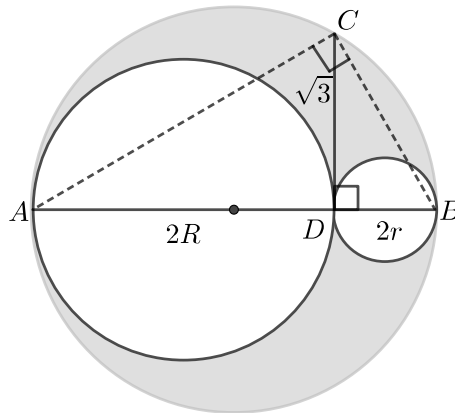
Solution

Join A to C and C to B . Since AB is a diameter and $\angle ACB$ is inscribed in a circle by that diameter, therefore $\angle ACB = 90^\circ$.

Let the radius of the smaller inside circle be r . Then the diameter of the smaller inside circle is $DB = 2r$. Let the radius of the larger inside circle be R . Then the diameter of the larger inside circle is $AD = 2R$.

Since $CD \perp AB$, then $\angle ADC = \angle BDC = 90^\circ$. We will use the Pythagorean Theorem in the three triangles $\triangle ADC$, $\triangle BDC$, and $\triangle ACB$, to establish a relationship between R and r .

All the information is marked in the following diagram.



$$\text{In } \triangle ADC, AC^2 = AD^2 + CD^2 = (2R)^2 + (\sqrt{3})^2 = 4R^2 + 3.$$

$$\text{In } \triangle BDC, CB^2 = BD^2 + CD^2 = (2r)^2 + (\sqrt{3})^2 = 4r^2 + 3.$$

$$\text{In } \triangle ACB, AB^2 = BC^2 + AC^2 = (4R^2 + 3) + (4r^2 + 3) = 4R^2 + 4r^2 + 6.$$

$$\text{But } AB^2 = (AD + DB)^2 = (2R + 2r)^2 = (2R + 2r)(2R + 2r) = 4R^2 + 8Rr + 4r^2$$

$$\therefore 4R^2 + 8Rr + 4r^2 = 4R^2 + 4r^2 + 6 \text{ and } 8Rr = 6 \text{ or } Rr = \frac{3}{4} \text{ follows.}$$





The radius of the smaller inner circle is r , the radius of the larger inner circle is R , and the radius of the outer circle is $(R + r)$. We can now find the shaded area.

$$\begin{aligned}
 \text{Shaded Area} &= \text{Area Outer Circle} - \text{Area Larger Inner Circle} - \text{Area Smaller Inner Circle} \\
 &= \pi \times (R + r)^2 - \pi \times R^2 - \pi \times r^2 \\
 &= \pi \times (R^2 + 2Rr + r^2) - \pi R^2 - \pi r^2 \\
 &= \pi R^2 + 2\pi Rr + \pi r^2 - \pi R^2 - \pi r^2 \\
 &= 2\pi Rr \\
 &= 2\pi \times \frac{3}{4}, \text{ since } Rr = \frac{3}{4} \\
 &= \frac{3\pi}{2}
 \end{aligned}$$

Therefore, the shaded area is $\frac{3\pi}{2}$ units².

NOTE: The relationship $Rr = \frac{3}{4}$ could also be established using similar triangles as follows:

In $\triangle ACD$, $\angle CAD + \angle ACD = 90^\circ$ (1).

Since $\angle ACB = 90^\circ$, $\angle ACD + \angle DCB = 90^\circ$ (2).

Subtracting (2) from (1) we get $\angle CAD - \angle DCB = 0$. The equation simplifies to $\angle CAD = \angle DCB$.

Now $\angle CAD = \angle DCB$ and $\angle CDA = \angle CDB = 90^\circ$. Therefore, $\triangle ADC \sim \triangle CDB$. From triangle similarity,

$$\begin{aligned}
 \frac{AD}{CD} &= \frac{CD}{DB} \\
 \frac{2R}{\sqrt{3}} &= \frac{\sqrt{3}}{2r} \\
 4Rr &= 3 \\
 Rr &= \frac{3}{4}
 \end{aligned}$$

