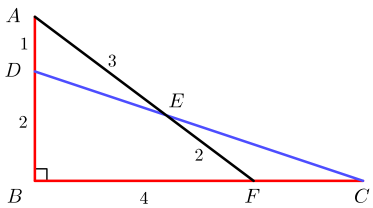


Problem of the Week

Problem E and Solution

Overlapping, Right?

Problem

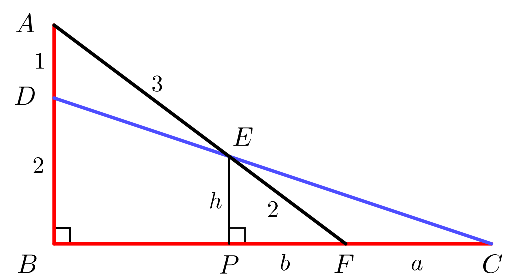
In the diagram, AB and BC are straight line segments meeting at B so that $\angle ABC = 90^\circ$. D lies on AB, F lies on BC and E is the intersection of AF and DC. Also, $AD = 1,\ DB = 2,\ AE = 3,\ BF = 4$ and $EF = 2$. Determine the length of CF.

Solution

Draw a perpendicular from E to BF.

Let P be the point where the perpendicular intersects BF.
Let $CF = a$, $PF = b$ and $EP = h$.

We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles to determine CF. The second solution uses trigonometry. The third solution is totally different.



Solution 1

Since EP is perpendicular to BC, we know $\angle EPF = 90^\circ$. Also, $\angle EFP = \angle AFB$ (same angle). Therefore, $\triangle ABF \sim \triangle EPF$ (by angle-angle triangle similarity).

From the similarity, $\dfrac{AF}{BF} = \dfrac{EF}{PF}$, so $\dfrac{5}{4} = \dfrac{2}{b}$ or $b = \dfrac{8}{5}$. Also, $\dfrac{AF}{AB} = \dfrac{EF}{EP}$, so $\dfrac{5}{3} = \dfrac{2}{h}$ or $h = \dfrac{6}{5}$.

Now let's calculate PC. We know $\angle EPC = \angle DBC = 90^\circ$ and $\angle ECP = \angle DCB$ (same angle).

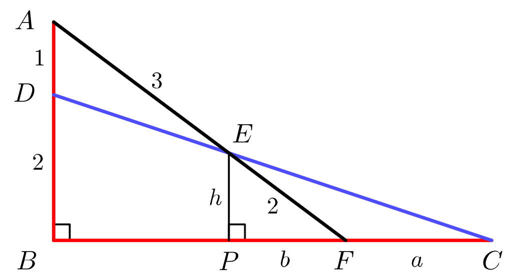
Therefore, $\triangle DBC \sim \triangle EPC$ (by angle-angle triangle similarity). This tells us $\dfrac{DB}{BC} = \dfrac{EP}{PC}$.

Since $BC = BF + CF = 4 + a$ and $PC = PF + CF = \frac{8}{5} + a$, we have

$$\begin{aligned}
\frac{DB}{BC} &= \frac{EP}{PC}\\
\frac{2}{4+a} &= \frac{\frac{6}{5}}{\frac{8}{5}+a}\\
\frac{16}{5} + 2a &= \frac{24}{5} + \frac{6}{5}a\\
2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5}\\
\frac{4}{5}a &= \frac{8}{5}\\
a &= 2
\end{aligned}$$

Therefore, $CF = 2$.

Solution 2



In $\triangle EPF$, $\sin \angle EFP = \frac{h}{2}$.

In $\triangle ABF$, $\sin \angle AFB = \frac{3}{5}$.

Since $\angle AFB = \angle EFP$ (same angle),

$$\begin{aligned} \sin \angle AFB &= \sin \angle EFP \\ \frac{3}{5} &= \frac{h}{2} \\ h &= \frac{6}{5} \end{aligned}$$

Since $\triangle EFP$ is a right-angled triangle,

$$\begin{aligned} EP^2 + PF^2 &= EF^2 \\ h^2 + b^2 &= 2^2 \\ \left(\frac{6}{5}\right)^2 + b^2 &= 4 \\ b^2 &= 4 - \frac{36}{25} \\ b^2 &= \frac{64}{25} \\ b &= \frac{8}{5} \quad \text{since} \quad b > 0 \end{aligned}$$

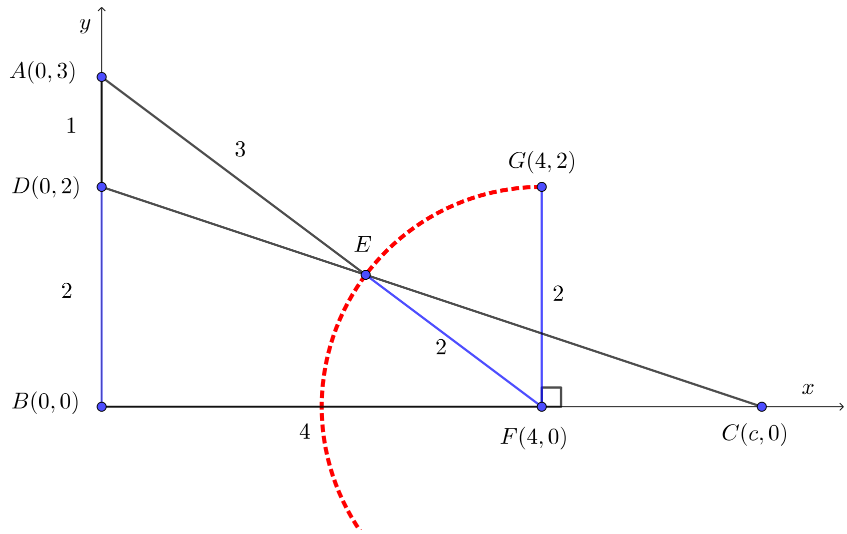
In $\triangle ECP$, $\tan \angle ECP = \frac{EP}{PC} = \frac{h}{a+b} = \frac{\frac{6}{5}}{a + \frac{8}{5}}$ and in $\triangle BCD$, $\tan \angle DCB = \frac{DB}{BC} = \frac{2}{4+a}$.

Since $\angle ECP = \angle DCB$ (same angle),

$$\begin{aligned} \tan \angle ECP &= \tan \angle DCB \\ \frac{\frac{6}{5}}{a + \frac{8}{5}} &= \frac{2}{4+a} \\ \frac{24}{5} + \frac{6}{5}a &= 2a + \frac{16}{5} \\ 2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5} \\ \frac{4}{5}a &= \frac{8}{5} \\ a &= 2 \end{aligned}$$

Therefore, $CF = 2$.

Solution 3



Using the given information, position B at the origin, D at $(0, 2)$ since $BD = 2$, A at $(0, 3)$ since $AD = 1$, F at $(4, 0)$ since $BF = 4$, and C on the positive x-axis at $(c, 0)$ with $c > 4$. Also note that E is in the first quadrant so $x > 0$ and $y > 0$. Construct $FG \perp BF$ with G at $(4, 2)$ so that $FG = FE = BD = 2$. Since $FG = FE = 2$, we can construct a circle with radius 2, centre $F(4, 0)$ and equation $(x - 4)^2 + y^2 = 4$. This circle intersects the line through AF at E.

The line passing through A and F has y-intercept 3 and slope $-\frac{3}{4}$, leading to equation $y = -\frac{3}{4}x + 3$. To find the intersection E, substitute $y = -\frac{3}{4}x + 3$ for y in $(x - 4)^2 + y^2 = 4$.

$$\begin{aligned}
(x-4)^2 + \left(-\frac{3}{4}x + 3\right)^2 &= 4 \\
x^2 - 8x + 16 + \frac{9}{16}x^2 - \frac{9}{2}x + 9 &= 4 \quad \text{(Expand the left side.)} \\
16x^2 - 128x + 256 + 9x^2 - 72x + 144 &= 64 \quad \text{(Multiply by 16.)} \\
25x^2 - 200x + 336 &= 0 \quad \text{(Simplify.)} \\
(5x - 12)(5x - 28) &= 0 \quad \text{(Factor.)}
\end{aligned}$$

It follows that $x = \frac{12}{5}$ or $x = \frac{28}{5}$. Substituting $x = \frac{12}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = \frac{6}{5}$. Substituting $x = \frac{28}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = -\frac{6}{5}$. But E is in the first quadrant so $y > 0$ and this second possibility is inadmissible. It follows that E has coordinates $\left(\frac{12}{5}, \frac{6}{5}\right)$.

We can now find the equation of the line containing D, E and C. This line has y-intercept 2, slope $-\frac{1}{3}$ and equation $y = -\frac{1}{3}x + 2$. The point C is the x-intercept of this line and is found by setting $y = 0$. This leads to x-intercept 6 and the point C has coordinates $(6, 0)$. Since F is at $(4, 0)$ and C is at $(6, 0)$, $CF = 2$. (It turns out that C is also on the circle through E and G.)