



Problem of the Week

Problem E and Solution

Check Your Units

Problem

The sum of the first n positive integers is $1 + 2 + 3 + \dots + n$. We define a_n to be the units digit of the sum of the first n positive integers.

Determine the smallest value of n such that $a_1 + a_2 + a_3 + \dots + a_n \geq 2019$.

Solution

Let's start by examining the values of a_n until we start to see a pattern.

$$\begin{aligned}a_1 &= 1 \\a_2 &= 3, \text{ since } 1 + 2 = 3 \\a_3 &= 6, \text{ since } 1 + 2 + 3 = 6 \\a_4 &= 0, \text{ since } 1 + 2 + 3 + 4 = 10 \\a_5 &= 5, \text{ since } 10 + 5 = 15 \\a_6 &= 1, \text{ since } 15 + 6 = 21\end{aligned}$$

Unfortunately, we do not have a pattern yet. Since $21 + 7 = 28$, $a_7 = 8$. We need to keep calculating values of a_n .

Notice that we can determine the units digit of the sum of the first n integers from the units digit from the sum of the first $n - 1$ integers and the units digit of n . For example, to calculate a_7 , we simply need to know that $a_6 = 1$ and the sum $1 + 7 = 8$ has units digit 8.

$$\begin{aligned}a_7 &= 8 \\a_8 &= 6, \text{ since } a_7 + 8 = 16 \\a_9 &= 5, \text{ since } a_8 + 9 = 15 \\a_{10} &= 5, \text{ since } a_9 + 0 = 5 \\a_{11} &= 6, \text{ since } a_{10} + 1 = 6 \\a_{12} &= 8, \text{ since } a_{11} + 2 = 8 \\a_{13} &= 1, \text{ since } a_{12} + 3 = 11 \\a_{14} &= 5, \text{ since } a_{13} + 4 = 5 \\a_{15} &= 0, \text{ since } a_{14} + 5 = 10 \\a_{16} &= 6, \text{ since } a_{15} + 6 = 6 \\a_{17} &= 3, \text{ since } a_{16} + 7 = 13 \\a_{18} &= 1, \text{ since } a_{17} + 8 = 11 \\a_{19} &= 0, \text{ since } a_{18} + 9 = 10 \\a_{20} &= 0, \text{ since } a_{19} + 0 = 0 \\a_{21} &= 1, \text{ since } a_{20} + 1 = 1\end{aligned}$$





The values of a_n should repeat now. Can you see why?

Since $a_{21} = a_1$ and the units digit of 22 equals the units digit of 2, $a_{22} = a_2$. Similarly, since $a_{22} = a_2$ and the units digit of 23 equals the units digit of 3, $a_{23} = a_3$.

We will also have $a_{24} = a_4$, and so on.

Therefore, the values of a_n will repeat every 20 values of n .

We can calculate

$$\begin{aligned} & a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + \\ & a_{16} + a_{17} + a_{18} + a_{19} + a_{20} \\ &= 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 + 6 + 8 + 1 + 5 + 0 + 6 + 3 + 1 + 0 + 0 \\ &= 70 \end{aligned}$$

Since the values of a_n repeat every 20 values of n , it is also true that

$$\begin{aligned} & a_{21} + a_{22} + a_{23} + \dots + a_{39} + a_{40} = 70, \text{ and} \\ & a_{41} + a_{42} + a_{43} + \dots + a_{59} + a_{60} = 70, \text{ and so on.} \end{aligned}$$

Since $\frac{2019}{70} = 28\frac{59}{70}$, there are 28 complete cycles of the 20 repeating values of a_n .

Therefore, the sum of the first $28 \times 20 = 560$ values of a_n sum to $28 \times 70 = 1960$. In other words, $a_1 + a_2 + a_3 + \dots + a_{559} + a_{560} = 1960$.

Let's keep adding values of a_n until we reach 2019.

$$\begin{aligned} & a_{561} + a_{562} + a_{563} + a_{564} + a_{565} + a_{566} + a_{567} + a_{568} + a_{569} + a_{570} \\ &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} \\ &= 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 \\ &= 40. \end{aligned}$$

Therefore, $a_1 + a_2 + a_3 \dots a_{569} + a_{570} = 1960 + 40 = 2000$.

$$a_{571} = a_{11} = 6, \quad a_{572} = a_{12} = 8, \quad a_{573} = a_{13} = 1 \text{ and } a_{574} = a_{14} = 5.$$

Now,

$$a_1 + a_2 + a_3 + \dots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} = 2000 + 6 + 8 + 1 = 2015 \leq 2019.$$

and

$$a_1 + a_2 + a_3 + \dots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} = 2015 + 5 = 2020 \geq 2019.$$

Therefore, the smallest value of n such that $a_1 + a_2 + a_3 + \dots + a_n \geq 2019$ is $n = 574$.

