Problem of the Week
Problem D and Solution
Perfect Squares

Problem

Determine the number of perfect squares less than 10 000 that are divisible by 392.

NOTE: A *perfect square* is an integer that can be expressed as the product of two equal integers. For example, 49 is a perfect square since $49 = 7 \times 7 = 7^2$.

Solution

In order to understand the nature of perfect squares, let's begin by examining the prime factorization of a few perfect squares.

From the example, $49 = 7^2$. Also, $36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2$, and $144 = 12^2 = (3 \times 4)^2 = 3^2 \times (2^2)^2 = 3^2 \times 2^4$.

From the above examples, we note that, for each perfect square, the exponent on each of its prime factors is an even integer greater than 0. This is because a perfect square is created by multiplying an integer by itself, so all of the primes in the factorization of the integer will appear twice. Also, for any integer a, if m is an even integer greater than or equal to zero, then a^m is a perfect square. This is because if m is an even integer greater than or equal to 0, then $m = 2n$ for some integer n greater than or equal to 0, and so $a^m = a^{2n} = a^n \times a^n$, where a^n is an integer.

To summarize, a positive integer is a perfect square exactly when the exponent on each prime in its prime factorization is even.

The number $392 = 8 \times 49 = 2^3 \times 7^2$. This is not a perfect square since the power 2^3 has an odd exponent. We require another factor of 2 to obtain a multiple of 392 that is a perfect square, namely $2 \times 392 = 784$. The number $784 = 2^4 \times 7^2 = (2^2 \times 7)^2 = 28^2$, and is the first perfect square less than 10 000 that is divisible by 392.

To find all the perfect squares less than 10 000 that are multiples of 392, we will multiply 784 by squares of positive integers, until we reach a product larger than 10 000.

If we multiply 784 by 2^2, we obtain 3136 which is 56^2, a second perfect square less than 10 000. If we multiply 784 by 3^2, we obtain 7056 which is 84^2, a third perfect square less than 10 000.

If we multiply 784 by 4^2, we obtain 12 544 which is a greater than 10 000. No other perfect squares divisible by 392 exist that are less than 10 000.

Therefore, there are 3 perfect squares less than 10 000 that are divisible by 392.