$w^x + y^z$

Problem of the Week

Problem E and Solution

Exponentially Large

Problem

Alex can choose four different numbers w, x, y and z from the set $\{-1, -2, -3, -4, -5\}$. What is the largest possible value of $w^x + y^z$?

Solution

Consider w^x and choose w and x to be different numbers from the set $\{-1, -2, -3, -4, -5\}$. What is the largest possible value for w^x?

Since x will be negative, we write $w^x = \dfrac{1}{w^{-x}}$, where $-x > 0$.

If x is odd, then since w is negative, then w^x will be negative.
If x is even, then w^x will be positive.
So to make w^x as large as possible, we make x even (ie. -2 or -4).

Also, in order to make $w^x = \dfrac{1}{w^{-x}}$ as large as possible, we want to make the denominator, w^{-x}, as small as possible, so w should be as small as possible in absolute value.

Therefore, the largest possible value of w^x will be when $w = -1$ and x is either -2 or -4, giving 1 in both cases (ie. $(-1)^{-2} = (-1)^{-4} = 1$).

What is the second largest possible value for w^x?
Again, we need x to be even to make w^x positive, and from above, we can assume that $w \neq -1$.
When $x = -2$, the smallest possible base (in absolute value) is $w = -3$ and $w^x = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$.

When $x = -4$, the smallest possible base (in absolute value) is $w = -2$ and $w^x = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$.

The largest of these two is $\dfrac{1}{9}$.

Therefore, the two largest possible values for w^x are 1 and $\dfrac{1}{9}$.

Thus, looking at $w^x + y^z$, since -1 can only be chosen for one of these four numbers, then the largest possible value for this expression is the sum of the largest two possible values for w^x, ie. $1 + \dfrac{1}{9} = \dfrac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$.

Therefore, the largest value of $w^x + y^z$ is $\dfrac{10}{9}$. (This will occur when $w = -1$, $x = -4$, $y = -3$, and $z = -2$ or $w = -3$, $x = -2$, $y = -1$, and $z = -4$.)