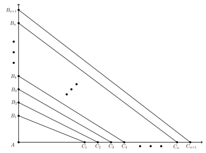
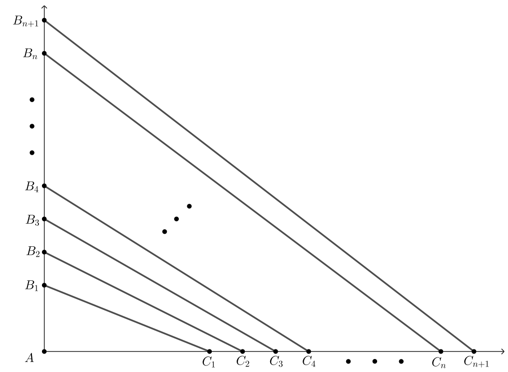


Problem of the Week

Problem E and Solution

The Area of the Year

Problem



In the diagram, $\triangle AB_1C_1$ is right-angled with $AB_1 = 2$ and $AC_1 = 5$. Lines AB_1 and AC_1 are extended and many more points are labelled at intervals of 1 unit, so that $B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = \cdots = 1$, and $C_1C_2 = C_2C_3 = C_3C_4 = C_4C_5 = \cdots = 1$. In fact, $B_1B_j = j-1$ and $C_1C_k = k-1$ for any positive integers j and k. For example, $B_1B_5 = 5-1 = 4$ and $C_1C_4 = 4-1 = 3$.
Determine the value of n so that the area of quadrilateral $B_nB_{n+1}C_{n+1}C_n$ is 2020.

Solution

Solution 1

In order to solve the problem, looking at the calculation of a specific area may prove helpful.
So let's determine the area of quadrilateral $B_4B_5C_5C_4$.

$$\begin{aligned}
\text{Area of quadrilateral } B_4B_5C_5C_4 &= \text{Area } \triangle B_5AC_5 - \text{Area } \triangle B_4AC_4 \\
&= \frac{1}{2}(AB_5)(AC_5) - \frac{1}{2}(AB_4)(AC_4) \\
&= \frac{1}{2}(2+(5-1))(5+(5-1)) - \frac{1}{2}(2+(4-1))(5+(4-1)) \\
&= \frac{1}{2}(6)(9) - \frac{1}{2}(5)(8) \\
&= 27 - 20 \\
&= 7 \text{ units}^2
\end{aligned}$$

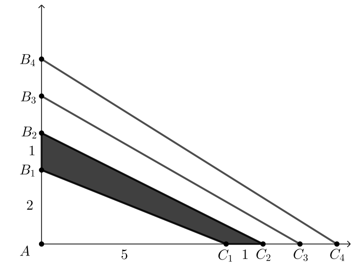
We will solve the problem by following what we did in the above example.

$$\begin{aligned}
\text{Area of quad. } B_nB_{n+1}C_{n+1}C_n &= \text{Area } \triangle B_{n+1}AC_{n+1} - \text{Area } \triangle B_nAC_n \\
2020 &= \frac{1}{2}(AB_{n+1})(AC_{n+1}) - \frac{1}{2}(AB_n)(AC_n) \\
2020 &= \frac{1}{2}(2+((n+1)-1))(5+((n+1)-1)) - \frac{1}{2}(2+(n-1))(5+(n-1)) \\
2020 &= \frac{1}{2}(2+n)(5+n) - \frac{1}{2}(1+n)(4+n) \\
4040 &= (2+n)(5+n) - (1+n)(4+n) \qquad \text{multiplying by 2} \\
4040 &= n^2+7n+10-(n^2+5n+4) \\
4040 &= n^2+7n+10-n^2-5n-4 \\
4040 &= 2n+6 \\
4034 &= 2n \\
2017 &= n
\end{aligned}$$

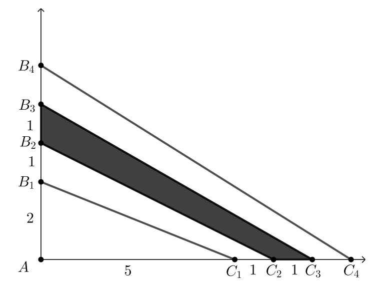
Therefore, the value of n is 2017.

Solution 2

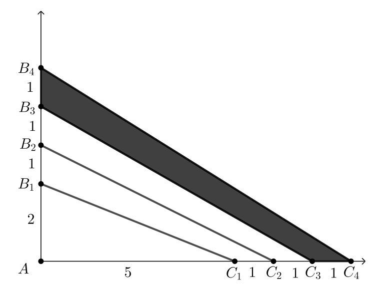
In this solution we look for a pattern in the area calculations.



$$\begin{aligned}
\text{Area of quad. } B_1B_2C_2C_1 &= \text{Area } \triangle B_2AC_2 - \text{Area } \triangle B_1AC_1\\
&= \frac{1}{2}(AB_2)(AC_2) - \frac{1}{2}(AB_1)(AC_1)\\
&= \frac{1}{2}(3)(6) - \frac{1}{2}(2)(5)\\
&= 9-5\\
\text{Area of first quad.} &= 4 \text{ units}^2
\end{aligned}$$



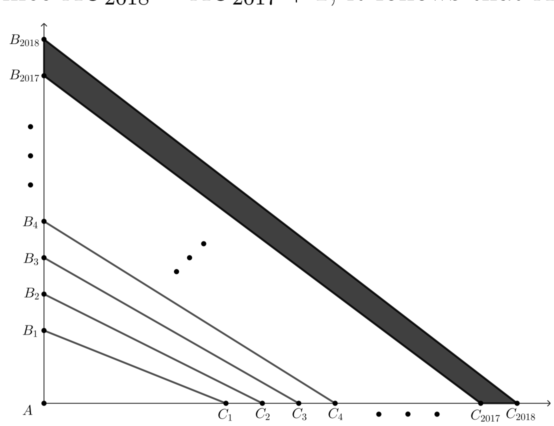
$$\begin{aligned}
\text{Area of quad. } B_2B_3C_3C_2 &= \text{Area } \triangle B_3AC_3 - \text{Area } \triangle B_2AC_2\\
&= \frac{1}{2}(AB_3)(AC_3) - \frac{1}{2}(AB_2)(AC_2)\\
&= \frac{1}{2}(4)(7) - \frac{1}{2}(3)(6)\\
&= 14-9\\
\text{Area of second quad.} &= 5 \text{ units}^2
\end{aligned}$$



$$\begin{aligned}
\text{Area of quad. } B_3B_4C_4C_3 &= \text{Area } \triangle B_4AC_4 - \text{Area } \triangle B_3AC_3\\
&= \frac{1}{2}(AB_4)(AC_4) - \frac{1}{2}(AB_3)(AC_3)\\
&= \frac{1}{2}(5)(8) - \frac{1}{2}(4)(7)\\
&= 20-14\\
\text{Area of third quad.} &= 6 \text{ units}^2
\end{aligned}$$

A possible pattern has emerged. The area of the quadrilateral appears to be three more than the position of the quadrilateral on the stack of consecutive quadrilaterals. If we want the area to be 2020, then it should be the 2017^{th} quadrilateral. That is, it should be the quadrilateral with vertices $B_{2017}B_{2018}C_{2018}C_{2017}$. Therefore, the value of n is 2017.

We can verify this value of n using the area calculation. Recall from the problem statement, $B_1B_j = j-1$ and $C_1C_k = k-1$ for any positive integers j and k.
So, $B_1B_{2017} = 2017-1 = 2016$. Then $AB_{2017} = AB_1 + B_1B_{2017} = 2+2016 = 2018$.
Since $AB_{2018} = AB_{2017}+1$, it follows that $AB_{2018} = 2019$.
Also, $C_1C_{2017} = 2017-1 = 2016$. Then $AC_{2017} = AC_1 + C_1C_{2017} = 5+2016 = 2021$.
Since $AC_{2018} = AC_{2017}+1$, it follows that $AC_{2018} = 2022$.



$$\begin{aligned}
&\text{Area of quadrilateral } B_{2017}B_{2018}C_{2018}C_{2017}\\
=\ &\text{Area } \triangle B_{2018}AC_{2018} - \text{Area } \triangle B_{2017}AC_{2017}\\
=\ &\frac{1}{2}(AB_{2018})(AC_{2018}) - \frac{1}{2}(AB_{2017})(AC_{2017})\\
=\ &\frac{1}{2}(2019)(2022) - \frac{1}{2}(2018)(2021)\\
=\ &2\,041\,209 - 2\,039\,189\\
=\ &2020 \text{ units}^2
\end{aligned}$$