Problem of the Week
Problem E and Solution
Terry's Triangles

Problem

Terry is drawing isosceles triangles with side lengths a, b, and c such that

$$a = y - x$$
$$b = x + z$$
$$c = y - z$$

Where x, y, and z are positive integers and $x + y + z < 10$.

Find all the possible triples (a, b, c) that satisfy this.

Solution

In an isosceles triangle, two sides must have equal length. So we need to consider three cases: $a = b$, $b = c$, and $a = c$. Also, in order for a, b, and c to represent side lengths of a triangle, they must be positive numbers and the sum of any two side lengths must be greater than the other side length.

Case 1: $a = b$

If $a = b$, then $y - x = x + z$, so $y = 2x + z$. We can make a table of all the values of x, y, and z that satisfy this equation as well as $x + y + z < 10$, and then find the corresponding values of a, b, and c and check if they are valid side lengths.

x	y	z	a	b	c	Valid?
1	3	1	2	2	2	Yes
1	4	2	3	3	2	Yes
1	5	3	4	4	2	Yes
2	5	1	3	3	4	Yes

Case 2: $b = c$

If $b = c$, then $x + z = y - z$, so $y = x + 2z$. As in Case 1, we can write the possible values of x, y, z, a, b, and c in a table.

x	y	z	a	b	c	Valid?
1	3	1	2	2	2	Yes
2	4	1	2	3	3	Yes
3	5	1	2	4	4	Yes
1	5	2	4	3	3	Yes

Case 3: $a = c$

If $a = c$, then $y - x = y - z$, so $x = z$. As in previous cases, we can write the possible values of x, y, z, a, b, and c in a table.

x	y	z	a	b	c	Valid?
1	1	1	0	2	0	No (a and c are not positive)
1	2	1	1	2	1	No ($a + c \not> b$)
1	3	1	2	2	2	Yes
1	4	1	3	2	3	Yes
1	5	1	4	2	4	Yes
1	6	1	5	2	5	Yes
1	7	1	6	2	6	Yes
2	1	2	-1	4	-1	No (a and c are not positive)
2	2	2	0	4	0	No (a and c are not positive)
2	3	2	1	4	1	No ($a + c \not> b$)
2	4	2	2	4	2	No ($a + c \not> b$)
2	5	2	3	4	3	Yes
3	1	3	-2	6	-2	No (a and c are not positive)
3	2	3	-1	6	-1	No (a and c are not positive)
3	3	3	0	6	0	No (a and c are not positive)
4	1	4	-3	8	-3	No (a and c are not positive)

Therefore, there are 12 possible triples (a, b, c). They are listed below.

$$\begin{array}{lllll} (2,2,2) & (3,3,2) & (4,4,2) & (3,3,4) & \\ (2,3,3) & (2,4,4) & (4,3,3) & & \\ (3,2,3) & (4,2,4) & (5,2,5) & (6,2,6) & (3,4,3) \end{array}$$