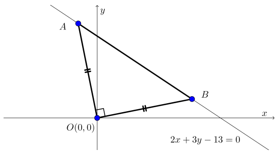



# Problem of the Week
## Problem E and Solution
## The Hypotenuse is Aligned

### Problem

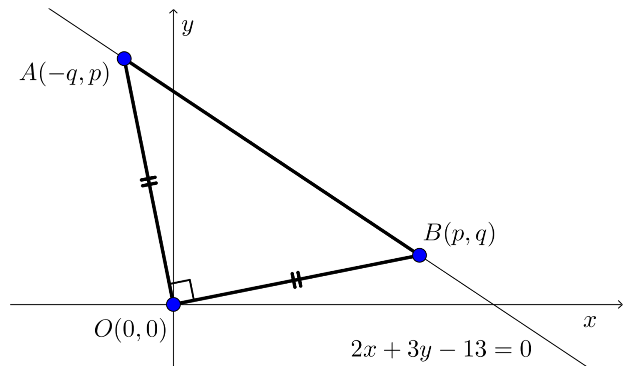
$\triangle OAB$ is an isosceles right-angled triangle with

- vertex $O$ located at the origin; and
- vertices $A$ and $B$ located on the line $2x + 3y - 13 = 0$ such that $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.

## Solution

### Solution 1




Let $B$ have coordinates $(p, q)$. Then the slope of $OB = \frac{q}{p}$. Since $\angle AOB = 90^\circ$, then $OB \perp OA$ and the slope of $OA$ is the negative reciprocal of the slope of $OB$. Therefore, the slope of $OA = \frac{p}{-q}$. Since the triangle is isosceles with $OA = OB$, it follows that the coordinates of $A$ are $(-q, p)$ . (We can verify this by finding the length of $OA$ and the length of $OB$ and showing that both lengths are equal to $\sqrt{p^2 + q^2}$.)

Since $B(p, q)$ lies on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.
Therefore, $2p + 3q - 13 = 0 \quad (1)$.

Since $A(-q, p)$ lies on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.
Therefore, $-2q + 3p - 13 = 0$, or $3p - 2q - 13 = 0 \quad (2)$.

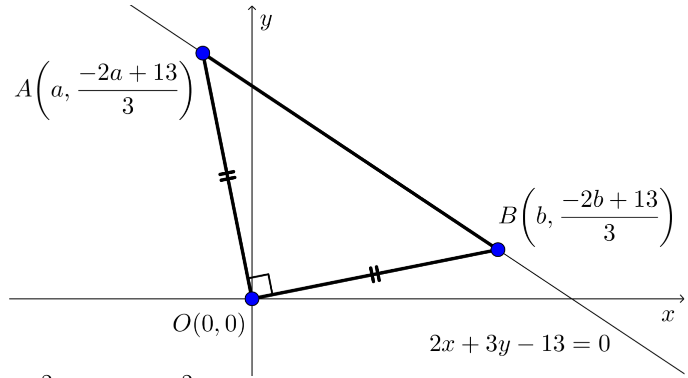
Since we have two equations and two unknowns, we can use elimination to solve for $p$ and $q$.

$$\begin{array}{rrcl}
(1) \times 2: & 4p + 6q - 26 & = & 0 \\
(2) \times 3: & 9p - 6q - 39 & = & 0 \\
\text{Adding, we obtain}: & 13p - 65 & = & 0 \\
 & p & = & 5 \\
\text{Substituting in } (1): & 10 + 3q - 13 & = & 0 \\
 & 3q & = & 3 \\
 & q & = & 1
\end{array}$$

Therefore, the point $B$ is $(5, 1)$ and the length of $OB$ is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.
$\triangle AOB$ is a right-angled triangle, so we can use $OB$ as the base and $OA$ as the height in the formula for the area of a triangle. Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26}\sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units$^2$.

## Solution 2




By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$. Since the points $A$ and $B$ are on the line, their coordinates satisfy the equation of the line. If $A$ has $x$-coordinate $a$, then $A$ has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If $B$ has $x$-coordinate $b$, then $B$ has coordinates $\left(b, \frac{-2b+13}{3}\right)$. Since $\triangle OAB$ is isosceles, we know that $OA = OB$ . Then

$$\begin{aligned}
OA^2 &= OB^2 \\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
\text{Multiplying by 9 :} \quad 9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
\text{Simplifying :} \quad 13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
\text{Rearranging :} \quad 13a^2 - 13b^2 - 52a + 52b &= 0 \\
\text{Dividing by 13 :} \quad a^2 - b^2 - 4a + 4b &= 0 \\
\text{Factoring pairs :} \quad (a+b)(a-b) - 4(a-b) &= 0 \\
\text{Common factoring :} \quad (a-b)(a+b-4) &= 0
\end{aligned}$$

Solving, $a = b$ or $a = 4 - b$. Since $A$ and $B$ are distinct points, $a \neq b$. Therefore, $a = 4 - b$. We can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4-b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4-b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is a right-angled triangle, we can use the Pythagorean Theorem, and $AB^2 = OA^2 + OB^2$ follows. But $OA = OB$, so this can be written $AB^2 = 2OB^2$.

$$\begin{aligned}
AB^2 &= 2OB^2 \\
(b-(4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2\left[b^2 + \left(\frac{-2b+13}{3}\right)^2\right] \\
(2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2\left[b^2 + \frac{4b^2 - 52b + 169}{9}\right] \\
4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} &= 2b^2 + \frac{8b^2 - 104b + 338}{9} \\
\text{Multiplying by 9 :} \quad 36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338 \\
\text{Simplifying :} \quad 52b^2 - 208b + 208 &= 26b^2 - 104b + 338 \\
\text{Rearranging :} \quad 26b^2 - 104b - 130 &= 0 \\
\text{Dividing by 26 :} \quad b^2 - 4b - 5 &= 0 \\
\text{Factoring :} \quad (b-5)(b+1) &= 0
\end{aligned}$$

It follows that $b = 5$ or $b = -1$. When $b = 5$, the point $A$ is $(-1, 5)$ and the point $B$ is $(5, 1)$. When $b = -1$, the point $A$ is $(5, 1)$ and the point $B$ is $(-1, 5)$. There are only two points. The area calculations shown in Solution 1 follow from here.

Therefore, the area of $\triangle OAB$ is 13 units$^2$.