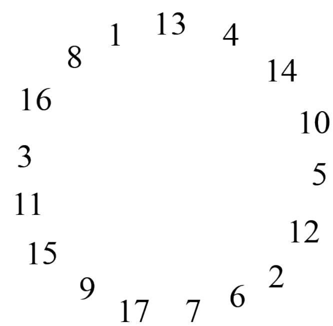
Problem of the Week

Problem E and Solution

Stand in a Circle

Problem

The numbers from 1 to 17 are arranged around a circle. One such arrangement is shown.



Explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

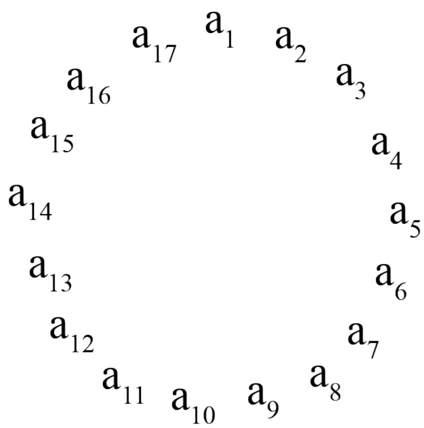
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution

We will use a *proof by contradiction* to explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

In general, to prove that a statement is true using a *proof by contradiction*, we first assume the statement is false. We then show this leads to a contradiction, which proves that our original assumption was wrong, and therefore the statement must be true.

First, we will assume that there exists an arrangement of the numbers from 1 to 17 around a circle where the sums of all groups of three adjacent numbers are less than 27. This arrangement is shown, where the variables $a_1,\ a_2,\ a_3, \ldots, a_{17}$ represent the numbers from 1 to 17, in some order, for this particular arrangement.



Now we will add up the sums of all groups of three adjacent numbers and call this value S.

$$\begin{aligned} S =& (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + (a_3 + a_4 + a_5) \\ &+ (a_4 + a_5 + a_6) + (a_5 + a_6 + a_7) + (a_6 + a_7 + a_8) \\ &+ (a_7 + a_8 + a_9) + (a_8 + a_9 + a_{10}) + (a_9 + a_{10} + a_{11}) \\ &+ (a_{10} + a_{11} + a_{12}) + (a_{11} + a_{12} + a_{13}) + (a_{12} + a_{13} + a_{14}) \\ &+ (a_{13} + a_{14} + a_{15}) + (a_{14} + a_{15} + a_{16}) + (a_{15} + a_{16} + a_{17}) \\ &+ (a_{16} + a_{17} + a_1) + (a_{17} + a_1 + a_2) \end{aligned}$$

We can see that there are 17 groups of three adjacent numbers around the circle. Since each of these groups has a sum that is less than 27, we can conclude that S must be less than $17 \times 27 = 459$. So, $S < 459$.

Looking again at the value of S, we can see that each of $a_1,\ a_2,\ a_3, \ldots, a_{17}$ appears exactly three times. So,

$$\begin{aligned} S &= 3(a_1) + 3(a_2) + 3(a_3) + \cdots + 3(a_{17}) \\ &= 3(a_1 + a_2 + a_3 + \cdots + a_{17}) \end{aligned}$$

However, we know that $a_1 + a_2 + a_3 + \cdots + a_{17}$ is equal to the sum of all the numbers from 1 to 17, which is $\frac{17(18)}{2} = 153$. Therefore, $S = 3(153) = 459$.

But this is a contradiction, since we stated earlier that $S < 459$. It can't be possible that $S < 459$ and $S = 459$. Therefore, our original assumption that there exists an arrangement of the numbers from 1 to 17 around a circle where the sums of all groups of three adjacent numbers are less than 27 must be false. Thus, it follows that every possible arrangement of the numbers from 1 to 17 around a circle must have at least one group of three adjacent numbers whose sum is at least 27.