Problem of the Week
Problem E and Solution
Red Dog

Problem

We can take any word and rearrange all the letters to get another "word". These new "words" may be nonsensical. For example, you can rearrange the letters in $MATH$ to get $MTHA$.

Nalan wants to rearrange all the letters in $REDDOG$. However, she uses the following rules:

- the letters R, E, and D cannot be adjacent to each other and in that order, and
- the letters D, O, and G cannot be adjacent to each other and in that order.

For example, the "words" $DOGRED$, $DDOGRE$, $GDREDO$, and $DREDOG$ are examples of unacceptable words in this problem, but $DROEGD$ is acceptable.

How many different arrangements of the letters in $REDDOG$ can Nalan make if she follows these rules?

Solution

We will find the total number of possible "words" Nalan can make, and then exclude those "words" which don't follow the rules (i.e. those which contain RED or DOG (or both)).

1. Determine the total number of "words" formed using 2 Ds, 1 E, 1 G, 1 O, and 1 R.

   First, place the E in 6 possible positions. Then, for each of the 6 possible placements of the E, there are 5 ways to place the G. There are then $6 \times 5 = 30$ ways to place the E and the G. Then, for each of the 30 possible placements of the E and G, there are 4 ways to place the O. There are then $30 \times 4 = 120$ ways to place the E, the G, and the O. Then, for each of the 120 possible placements of the E, G, and O, there are 3 ways to place the R. There are then $120 \times 3 = 360$ ways to place the E, the G, the O, and the R.

   For each of the 360 ways to place the E, G, O, and R, the 2 Ds must go in the remaining two empty spaces in 1 way. Therefore, there are $360 \times 1 = 360$ ways to place the E, the G, the O, the R, and the 2 Ds.

   Thus, there are 360 possible "words" that Nalan can make.

2. Determine how many "words" contain RED.

   There are 4 ways to place the word RED in the six spaces. The word RED could start in the first, second, third, or fourth position.

   $$R\ E\ D\ \_\ \_\ \_ \qquad \_\ R\ E\ D\ \_\ \_ \qquad \_\ \_\ R\ E\ D\ \_ \qquad \_\ \_\ \_\ R\ E\ D$$

   For each placement of the word RED, there are 6 ways to place the letters of the word DOG in the remaining three spaces: DOG, DGO, GDO, GOD, ODG and OGD. So there are $4 \times 6 = 24$ "words" containing RED.

3. Determine how many "words" contain DOG.

   There are 4 ways to place the word DOG in the six spaces. The word DOG could start in the first, second, third, or fourth position.

   $$D\ O\ G\ \_\ \_\ \_ \qquad \_\ D\ O\ G\ \_\ \_ \qquad \_\ \_\ D\ O\ G\ \_ \qquad \_\ \_\ \_\ D\ O\ G$$

   For each placement of the word DOG, there are 6 ways to place the letters of the word RED in the remaining three spaces: DER, DRE, EDR, ERD, RDE and RED. So there are $4 \times 6 = 24$ "words" containing DOG.

4. Determine how many "words" contain both RED and DOG.

   There are 4 "words" that contain both RED and DOG. They are as follows.

   $$REDDOG \qquad DOGRED \qquad REDOGD \qquad DREDOG$$

   These 4 "words" have been double-counted, as they would have been counted in both step 2 and step 3.

Thus in total, there are $24 + 24 - 4 = 44$ "words" that contain RED or DOG (or both). Since there are 360 possible "words" Nalan can make, we can subtract 44 from this to determine the number of these "words" that do not contain RED or DOG (or both).

Therefore, $360 - 44 = 316$ "words" can be formed in which the letters R, E and D are not adjacent to each other and in that order and the letters D, O and G are not adjacent to each other and in that order.