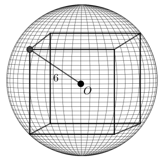


Problem of the Week

Problem E and Solution

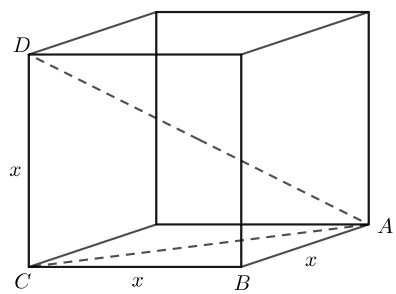
All Around the Cube

Problem

A cube is said to be *inscribed* in a sphere when all the vertices of the cube are on the surface of the sphere. In the diagram below, the cube is inscribed in the sphere with centre O. If the radius of the sphere is 6 cm, determine the volume of the cube.

Solution

Label four of the vertices of the cube A, B, C, D, as shown in the diagram. Let x represent the side length of the cube. Then $AB = BC = CD = x$.



The diagonals of a cube intersect at a point such that the distance from the intersection point to each vertex is equal. Since each vertex of the cube is on the sphere, the diagonal of the cube, AD, is equal in length to the diameter of the sphere. Therefore, $AD = 2(6) = 12$ cm.

Each face of a cube is a square, so $\angle ABC = 90^\circ$. Using the Pythagorean Theorem in $\triangle ABC$,

$$AC^2 = AB^2 + BC^2 = x^2 + x^2 = 2x^2$$

In a cube the sides are perpendicular to the base. In particular, DC is perpendicular to the base and it follows that $DC \perp AC$. Therefore $\triangle DCA$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle DCA$,

$$AD^2 = AC^2 + CD^2 = 2x^2 + x^2 = 3x^2$$

But $AD = 12$, so $AD^2 = 144$. Then,

$$\begin{aligned} 3x^2 &= 144 \\ x^2 &= 48 \\ x &= 4\sqrt{3}, \quad \text{since } x > 0 \end{aligned}$$

Therefore, the volume of the cube is $x^3 = (4\sqrt{3})^3 = 192\sqrt{3}$ cm^3.