Problem of the Week
Problem E and Solution
A Small Subset

Problem

There are 90 000 five-digit positive integers. However, only some of these five-digit integers satisfy the following conditions:

- the middle digit is 0,
- the ten thousands digit and the tens digit are equal,
- the thousands digit and the ones (units) digit are equal, and
- the number has exactly 5 prime factors. All of these prime factors are odd and none are repeated.

Determine all five-digit positive integers that satisfy the given conditions.

Solution

Solution 1

Let $st0st$ represent a five-digit positive integer satisfying the conditions. Notice that

$$st0st = st(1000) + st = st(1000 + 1) = st(1001)$$

This means that the number $st0st$ is divisible by 1001, which is the product of the three odd prime factors 7, 11, and 13. So st is a two-digit number which is the product of two different odd prime factors none of which can be 7, 11 or 13. We will now generate all possible two-digit products, using odd prime factors other than 7, 11 and 13.

Prime Factor a	Prime Factor b	st $= a \times b$	Five Different Odd Primes	Product $st0st$
3	5	15	$3, 5, 7, 11, 13$	15015
3	17	51	$3, 7, 11, 13, 17$	51051
3	19	57	$3, 7, 11, 13, 19$	57057
3	23	69	$3, 7, 11, 13, 23$	69069
3	29	87	$3, 7, 11, 13, 29$	87087
3	31	93	$3, 7, 11, 13, 31$	93093
5	17	85	$5, 7, 11, 13, 17$	85085
5	19	95	$5, 7, 11, 13, 19$	95095

No other two-digit product of two different odd prime factors other than 7, 11 and 13 exists.

Therefore, there are 8 five-digit positive integers that satisfy the given conditions. The numbers are 15015, 51051, 57057, 69069, 87087, 93093, 85085 and 95095.

Solution 2

Let $st0st$ represent a five-digit positive integer satisfying the conditions.

A number is divisible by 11 if the difference between the sum of the digits in the even positions and the sum of the digits in the odd positions is a multiple of 11. (This problem can still be done without knowing this divisibility fact but the task is made simpler with it.) The sum of the digits in the even positions is $st0st$ is $t + s$. The sum of the digits in the odd positions is $s + 0 + t$ which simplifies to $s + t$. The difference of the two sums is $(t + s) - (s + t) = 0$ which is a multiple of 11. Therefore, $st0st$ is divisible by 11.

Only odd factors are used, so the product will be odd. This means that the product looks like $s10s1$, $s30s3$, $s50s5$, $s70s7$, or $s90s9$ where s is a digit from 1 to 9. So we begin to systematically look at the possibilities.

First, we will examine numbers that have 3 (and 11) as a factor. To be divisible by 3, the sum of the digits will be divisible by 3. To be divisible by 9, the sum of the digits will be divisible by 9. But if the number is divisible by 9, it is divisible by 3^2 and would have a repeated prime factor which is not allowed. So we want numbers divisible by 3 but not 9. The possibilities are as follows: 21021, 51051, 33033, 93093, 15015, 75075, 57057, 87087, 39039, and 69069. The sum of the digits of each of these numbers is divisible by 3 so each of the numbers are divisible by 3. The numbers 81081, 63063, 45045, 27027 and 99099 are divisible by 9 and have therefore been eliminated.

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$21021 = 3 \times 11 \times 637 = 3 \times 11 \times 7 \times 91 = 3 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$51051 = 3 \times 11 \times 1547 = 3 \times 11 \times 7 \times 221 = 3 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 51051 is a **valid number**.

$$33033 = 3 \times 11 \times 1001 = 3 \times 11 \times 7 \times 143 = 3 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$93093 = 3 \times 11 \times 2821 = 3 \times 11 \times 7 \times 403 = 3 \times 11 \times 7 \times 13 \times 31$$

Since there are 5 different odd prime factors, 93093 is a **valid number**.

$$15015 = 3 \times 11 \times 455 = 3 \times 11 \times 5 \times 91 = 3 \times 11 \times 5 \times 7 \times 13$$

Since there are 5 different odd prime factors, 15015 is a **valid number**.

$$75075 = 3 \times 11 \times 2275 = 3 \times 11 \times 5 \times 455 = 3 \times 11 \times 5 \times 5 \times 91 = 3 \times 11 \times 5 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated and there are six prime factors, this is not a valid number.

$$57057 = 3 \times 11 \times 1729 = 3 \times 11 \times 7 \times 247 = 3 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 57057 is a **valid number**.

$$87087 = 3 \times 11 \times 2639 = 3 \times 11 \times 7 \times 377 = 3 \times 11 \times 7 \times 13 \times 29$$

Since there are 5 different odd prime factors, 87087 is a **valid number**.

$$39039 = 3 \times 11 \times 1183 = 3 \times 11 \times 7 \times 169 = 3 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$69069 = 3 \times 11 \times 2093 = 3 \times 11 \times 7 \times 299 = 3 \times 11 \times 7 \times 13 \times 23$$

Since there are 5 different odd prime factors, 69069 is a **valid number**.

Second, we will examine numbers that are divisible by 5 but not 3, since divisibility by 3 has been examined. If a number is divisible by 5, then it ends in 5 or 0. Since the number is odd, we can exclude any number ending in 0, leaving 25025, 35035, 55055, 65065, 85085 and 95095 as possible numbers. (15015, 45045, 75075 were examined above and have been excluded.)

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$25025 = 5 \times 11 \times 455 = 5 \times 11 \times 5 \times 91 = 5 \times 11 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated, this is not a valid number.

$$35035 = 5 \times 11 \times 637 = 5 \times 11 \times 7 \times 91 = 5 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$55055 = 5 \times 11 \times 1001 = 5 \times 11 \times 7 \times 143 = 5 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$65065 = 5 \times 11 \times 1183 = 5 \times 11 \times 7 \times 169 = 5 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$85085 = 5 \times 11 \times 1547 = 5 \times 11 \times 7 \times 221 = 5 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 85085 is a **valid number**.

$$95095 = 5 \times 11 \times 1729 = 5 \times 11 \times 7 \times 247 = 5 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 95095 is a **valid number**.

Thirdly, we will look at numbers that are divisible by 7 but not divisible by 3 or 5. If we multiply 7 by the next four odd prime numbers we get $7 \times 11 \times 13 \times 17 \times 19 = 323\,323$, a six-digit number, so we are beyond all possible solutions.

Therefore, there are 8 positive five-digit integers that satisfy the given conditions. The numbers are 51051, 93093, 15015, 57057, 87087, 69069, 85085 and 95095.