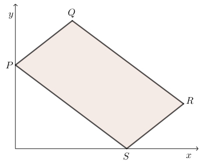


Problem of the Week
Problem E and Solution
Parallelogram Askew

Problem

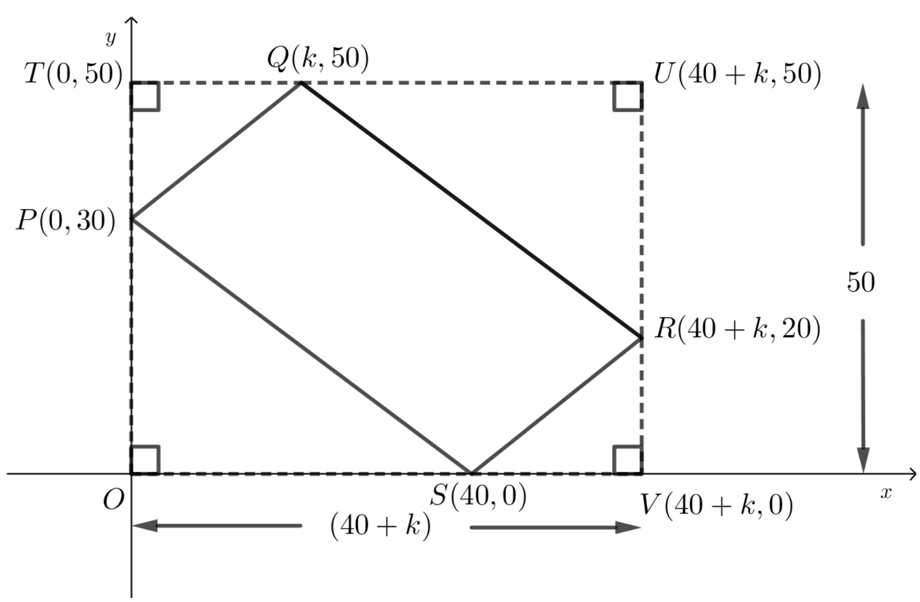
Parallelogram $PQRS$ is positioned such that P lies on the positive y-axis, S lies on the positive x-axis, and Q and R lie in the first quadrant. If vertices P, Q, and S are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively, and the area of $PQRS$ is 1340 units^2, determine the coordinates of Q and R.

Solution
Solution 1:

In this solution we will use a method known commonly as "*completing the rectangle*".

Since $PQRS$ is a parallelogram, $PQ = SR$ and PQ is parallel to SR. We can use this to find the coordinates of R. To get from P to Q, we go up 20 units and right k units. Therefore, to get from S to R we do the same. Therefore, R is located at $(40 + k, 20)$.

Enclose $PQRS$ in rectangle $OTUV$ such that OT is on the positive y-axis passing through P, TU is parallel to the positive x-axis passing through Q, UV is parallel to the positive y-axis passing through R, and OV lies along the positive x-axis passing through S. The y-coordinate of Q is the distance from the x-axis to TU and also the height, UV, of rectangle $OTUV$. It follows that $OT = UV = 50$ units. Therefore, the coordinates of T are $(0, 50)$. Similarly, the x-coordinate of R is the distance from the y-axis to UV and also the width, OV, of rectangle $OTUV$. It follows that $TU = OV = (40 + k)$ units. Therefore, the coordinates of V are $(40 + k, 0)$ and the coordinates of U are $(40 + k, 50)$.



We can now put the information together using areas to determine the value of k.

$$\begin{aligned}
\text{Area } OTUV &= \text{Area } \triangle PTQ + \text{Area } \triangle QUR + \text{Area } \triangle RVS + \text{Area } \triangle SOP + \text{Area } PQRS \\
UV \times OV &= \frac{PT \times TQ}{2} + \frac{QU \times UR}{2} + \frac{RV \times VS}{2} + \frac{OS \times OP}{2} + 1340 \\
50 \times (40 + k) &= \frac{(50-30) \times k}{2} + \frac{((40+k) - k) \times (50-20)}{2} + \frac{20 \times ((40+k) - 40)}{2} + \frac{40 \times 30}{2} + 1340 \\
50 \times (40 + k) &= \frac{20 \times k}{2} + \frac{40 \times 30}{2} + \frac{20 \times k}{2} + \frac{40 \times 30}{2} + 1340 \\
2000 + 50k &= 10k + 600 + 10k + 600 + 1340 \\
2000 + 50k &= 20k + 2540 \\
30k &= 540 \\
k &= 18
\end{aligned}$$

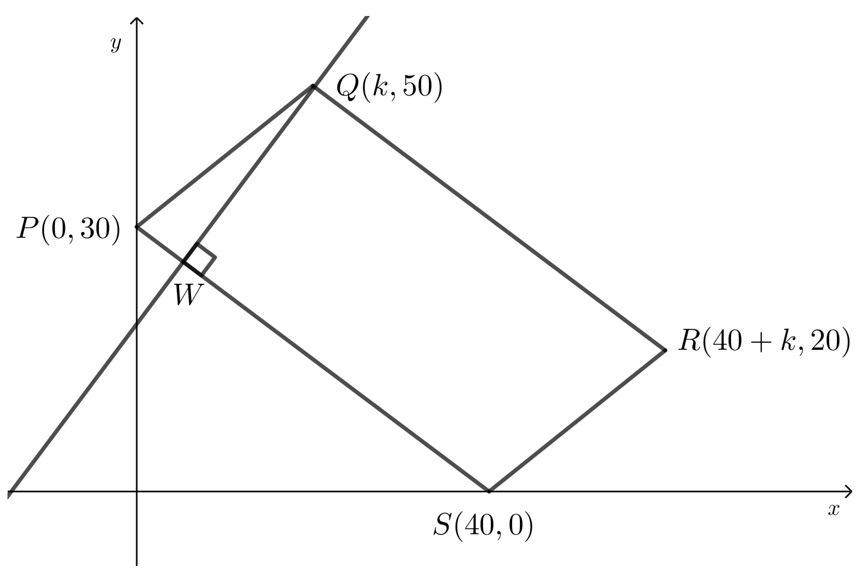
Therefore, the value of k is 18 and coordinates of Q and R are $Q(18, 50)$ and $R(58, 20)$.

Solution 2:

In this solution we will use linear equations, intersections, and lengths to find k.

Since $PQRS$ is a parallelogram, $PQ = SR$ and PQ is parallel to SR. We can use this to find the coordinates of R. To get from P to Q, we go up 20 units and right k units. Therefore, to get from S to R we do the same. Therefore, R is located at $(40 + k, 20)$.

Construct a line perpendicular to PS that passes through Q and meets PS at W.



We are going to find the coordinates of W in terms of k.

The line through PS has a slope of $-\frac{3}{4}$ and a y-intercept of 30. Therefore, the equation of this line is

$$y = -\frac{3}{4}x + 30 \tag{1}$$

The line though QW is perpendicular to PS and so has slope $\frac{4}{3}$. The equation of this line is $4x - 3y = C$. Substituting the coordinates of Q into this equation, we get $4k - 3(50) = C$ or $C = 4k - 150$. Therefore, the line through QW has equation

$$4x - 3y = 4k - 150 \qquad (2)$$

W is the intersection point of the lines with equations (1) and (2).

Substituting equation (1) into equation (2), we get:

$$\begin{aligned} 4x - 3\left(-\frac{3}{4}x + 30\right) &= 4k - 150 \\ 4x + \frac{9}{4}x - 90 &= 4k - 150 \\ 16x + 9x - 360 &= 16k - 600 \\ 25x &= 16k - 240 \\ x &= 0.64k - 9.6 \qquad (3) \end{aligned}$$

Substituting equation (3) into equation (1) we get:

$$\begin{aligned} y &= -\frac{3}{4}(0.64k - 9.6) + 30 \\ y &= -0.48k + 37.2 \end{aligned}$$

Therefore, the point W has coordinates $(0.64k - 9.6, -0.48k + 37.2)$.

We will now find two expressions for the length of QW.

Using the distance formula we know

$$QW = \sqrt{(0.64k - 9.6 - k)^2 + (-0.48k + 37.2 - 50)^2} \qquad (4)$$

Another way to find the length QW is using the area of the parallelogram.

The length of $PS = \sqrt{(30-0)^2 + (0-40)^2} = \sqrt{2500} = 50$, since $PS > 0$.

PS is the base of the parallelogram and QW is the height. Therefore,

$$\begin{aligned} PS \times QW &= 1340 \\ 50QW &= 1340 \\ QW &= 26.8 \qquad (5) \end{aligned}$$

Now equating equations (4) and (5), we can solve for k.

$$\begin{aligned} \sqrt{(0.64k - 9.6 - k)^2 + (-0.48k + 37.2 - 50)^2} &= 26.8 \\ (-0.36k - 9.6)^2 + (-0.48k - 12.8)^2 &= 718.24 \\ 0.1296k^2 + 6.912k + 92.16 + 0.2034k^2 + 12.288k + 163.84 &= 718.24 \\ 0.36k^2 + 19.2k - 462.24 &= 0 \end{aligned}$$

Using the quadratic formula, we find $k = 18$ or $k = -\frac{214}{3}$. Since $Q(k, 50)$ is in the first quadrant, we must have $k > 0$ and so $k = 18$.

Therefore, the coordinates of Q and R are $Q(18, 50)$ and $R(58, 20)$.