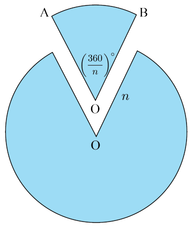

Problem of the Week

Problem D and Solution

One Slice at a Time

Problem

Points A and B are on a circle with centre O and radius n so that $\angle AOB = \left(\frac{360}{n}\right)^\circ$. Sector AOB is cut out of the circle. Determine all positive integers n for which the perimeter of sector AOB is greater than 20 and less than 30.

NOTE: You may use the fact that the ratio of the length of an arc to the circumference of the circle is the same as the ratio of the sector angle to 360°. In fact, the same ratio holds when comparing the area of a sector to the total area of the circle.

Solution

In general, as the sector angle gets larger, so does the length of the arc, if the radius remains the same. However in this problem, as the radius n increases, the sector angle $\left(\frac{360}{n}\right)^\circ$ decreases. So it is difficult to "see" what happens to the length of the arc.

We know the ratio of the arc length to the circumference of the circle is the same as the ratio of the sector angle to 360°. That is,

$$\frac{\text{arc length of } AB}{\text{circumference}} = \frac{\text{sector angle of } AOB}{360^\circ}$$

Rearranging, we have

$$\text{arc length of } AB = \frac{\text{sector angle of } AOB}{360^\circ} \times \text{circumference}$$

We know circumference $= \pi d = \pi \times 2n$, since $d = 2n$. Thus,

$$\text{arc length of } AB = \frac{\frac{360}{n}}{360} \times \pi \times 2n = 2\pi$$

Now we can use the arc length to calculate the perimeter of AOB.

$$\begin{aligned}\text{perimeter of } AOB &= AO + OB + \text{arc length of } AB\\ &= n + n + 2\pi\\ &= 2n + 2\pi\end{aligned}$$

If the perimeter is greater than 20, then

$$\begin{aligned}2n + 2\pi &> 20\\ n + \pi &> 10\\ n &> 10 - \pi \approx 6.9\end{aligned}$$

If the perimeter is less than 30, then

$$\begin{aligned}2n + 2\pi &< 30\\ n + \pi &< 15\\ n &< 15 - \pi \approx 11.9\end{aligned}$$

We want all integer values of n such that $n > 6.9$ and $n < 11.9$. The only integer values of n that satisfy these conditions are $n = 7$, $n = 8$, $n = 9$, $n = 10$, and $n = 11$.