Problem of the Week
Problem D and Solution
Sharing Sweets

Problem

Diana has 10 candies, and she wishes to give all 10 candies to her three friends, Victoria, Manuela, and Alejandra. She does not necessarily want to distribute the candy equally, but she does want each friend to receive at least one candy. In how many ways can she distribute the candies to Victoria, Manuela, and Alejandra?

Solution

We know that there are 10 candies and that each friend must receive at least one. We will consider the following cases:

1. Victoria receives one candy. Then Manuela and Alejandra receive a total of $10 - 1 = 9$ candies between them. This can be done in 8 possible ways:
$$(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)$$

2. Victoria receives two candies. Then Manuela and Alejandra receive a total of $10 - 2 = 8$ candies between them. This can be done in 7 possible ways:
$$(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1)$$

3. Victoria receives three candies. Then Manuela and Alejandra receive a total of $10 - 3 = 7$ candies between them. This can be done in 6 possible ways:
$$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$$

4. Victoria receives four candies. Then Manuela and Alejandra receive a total of $10 - 4 = 6$ candies between them. This can be done in 5 possible ways:
$$(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$$

5. Victoria receives five candies. Then Manuela and Alejandra receive a total of $10 - 5 = 5$ candies between them. This can be done in 4 possible ways: $(1, 4), (2, 3), (3, 2), (4, 1)$.

6. Victoria receives six candies. Then Manuela and Alejandra receive a total of $10 - 6 = 4$ candies between them. This can be done in 3 possible ways: $(1, 3), (2, 2), (3, 1)$.

7. Victoria receives seven candies. Then Manuela and Alejandra receive a total of $10 - 7 = 3$ candies between them. This can be done in 2 possible ways: $(1, 2), (2, 1)$.

8. Victoria receives eight candies. Then Manuela and Alejandra receive a total of $10 - 8 = 2$ candies between them. This can be done in 1 way: $(1, 1)$.

Notice that Victoria cannot receive more than eight candies. If she does, then at least one of Manuela and Alejandra would have received zero candies.

Thus, the total number of ways Diana can distribute 10 candies between the three friends so that each receives at least one candy is $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ ways. This sum can be computed by adding the positive integers from 1 to 8. However, it is also known that the sum of the first n positive integers can be calculated using the formula
$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$ In this case $n = 8$, so the sum is $\frac{8(9)}{2} = 36$.