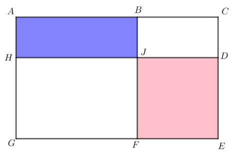


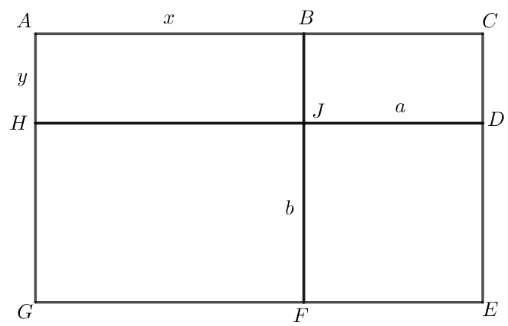
Problem of the Week
Problem D and Solution
Find the Largest Area

Problem

Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$.

Solution



Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.
Then,

$$HJ = GF = AB = x$$
$$BJ = CD = AH = y$$
$$BC = FE = JD = a$$
$$HG = DE = JF = b$$

Thus, we have

$$\begin{aligned}\text{area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG) \\ &= 6 + ya + 15 + xb \\ &= 21 + ya + xb\end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

To maximize the area, we need to pick the values of x, y, a, and b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

- **Case 1:** $x = 1$ cm, $y = 6$ cm, $a = 1$ cm, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42$ cm^2.
- **Case 2:** $x = 1$ cm, $y = 6$ cm, $a = 15$ cm, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112$ cm^2.
- **Case 3:** $x = 1$ cm, $y = 6$ cm, $a = 3$ cm, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44$ cm^2.
- **Case 4:** $x = 1$ cm, $y = 6$ cm, $a = 5$ cm, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54$ cm^2.
- **Case 5:** $x = 2$ cm, $y = 3$ cm, $a = 1$, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54$ cm^2.
- **Case 6:** $x = 2$ cm, $y = 3$ cm, $a = 15$, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68$ cm^2.
- **Case 7:** $x = 2$ cm, $y = 3$ cm, $a = 3$, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40$ cm^2.
- **Case 8:** $x = 2$ cm, $y = 3$ cm, $a = 5$, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42$ cm^2.

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram given in the problem was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.