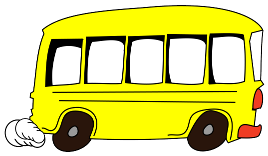

Problem of the Week

Problem D and Solution

Teacher Road Trip 2

Problem

To help pass time on a long bus ride, a group of math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first and second teachers each said a non-negative integer, and every teacher after that said the sum of all of the previous terms in the sequence.

For example, if the first teacher said the number 2 and the second teacher said the number 8, then

- the third teacher would say the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher would say the sum of the first, second, and third terms, which is $2 + 8 + 10 = 20$.

How many possible sequences could the teachers have said if the first teacher said the number 3 and another teacher said the number 3072?

Solution

We know how to construct the sequence, and we know that the first term is 3, but where is the term whose value is 3072?

- Could 3072 be the second term?

  If the first two terms are 3 and 3072, then we can calculate the next few terms.

  - The third term would be $3 + 3072 = 3075$.
  - The fourth term would be $3 + 3072 + 3075 = 3075 + 3075 = 2(3075) = 6150$.
  - The fifth term would be $3 + 3072 + 3075 + 6150 = 6150 + 6150 = 2(6150) = 12\,300$.

  We see that we can determine any term beyond the third term by summing all of the previous terms, or we can simply double the term immediately before the required term, since that term is the sum of all the preceding terms. (This also means that if any term after the third term is known, then the preceding term is half the value of that term.)

  Therefore, there is one possible sequence with 3072 as the second term. The first 6 terms of this sequence are $3,\ 3072,\ 3075,\ 6150,\ 12\,300,\ 24\,600$.

- Could 3072 be the third term?

  Yes, since the third term is the sum of the first two terms, and the first term is 3, then the second term would be $3072 - 3 = 3069$ and the first 6 terms of this sequence are $3,\ 3069,\ 3072,\ 6144,\ 12\,288,\ 24\,576$.

- Could 3072 be the fourth term?

  Yes, since the fourth term is even, then we can determine the third term to be half of the fourth term, which is $3072 \div 2 = 1536$, then the second term would be $1536 - 3 = 1533$. The first 6 terms of this sequence are $3,\ 1533,\ 1536,\ 3072,\ 6144,\ 12\,288$.

- Could 3072 be the fifth term?

  To get from the fifth term to the third term we would divide by 2 twice, or we could divide by 4. If the resulting third term is a non-negative integer greater than or equal to 3, then the sequence exists. The third term would be $3072 \div 4 = 768$, and the second term would be $768 - 3 = 765$. Thus the sequence exists and the first 6 terms are $3,\ 765,\ 768,\ 1536,\ 3072,\ 6144$.

We could continue in this way until we discover all possible sequences that are formed according to the given rules with first term 3 and 3072 somewhere in the sequence. However, if we look at the prime factorization of 3072 we see that the highest power of 2 that divides 3072 is 1024 (or 2^{10}), since $3072 = 2^{10} \times 3$. In fact, dividing 3072 by 1024 would produce a third term that would be 3. The second term would then be 0, a non-negative integer, and the resulting sequence would be $3,\ 0,\ 3,\ 6,\ 12,\ 24,\ 48,\ 96,\ 192,\ 384,\ 768,\ 1536,\ 3072,\ 6144, \ldots$.

If we divide 3072 by any integral power of 2 from $2^0 = 1$ to $2^{10} = 1024$, the resulting third term would be an integer greater than or equal to 3, and 3072 would appear in each of these sequences. There are 11 such sequences. The number 3072 would appear somewhere from term 3 to term 13 in the acceptable sequence. However, 3072 can also appear as the second term, so there are a total of 12 possible sequences.

Could 3072 be the fourteenth term? From the fourteenth term to the third term we would need to divide 3072 by 2^{11}. The resulting third term would be $\frac{3}{2}$. This would mean the second term is not an integer and so the sequence is not possible.

Therefore, there are a total of 12 such sequences.