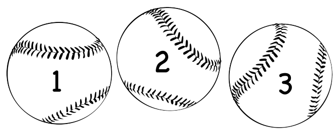

Problem of the Week
Problem D and Solution
The Baseball Game

Problem

Ivy has created a game for her school's math fair. She put three baseballs, numbered 1, 2, and 3, into a bag. Without looking, a player will randomly draw a baseball from the bag, record its number, and then put the baseball back into the bag. They will do this two more times and then calculate the sum of the three numbers recorded. If the sum is less than 8, the player will win a prize.

What is the probability that a player will win a prize when they play this game once?

Solution

In order to determine the probability, we must determine the number of ways three baseballs whose sum is less than 8 can be drawn from the bag, and then divide by the total number of ways three baseballs can be drawn from the bag.

First, let's determine the total number of ways three baseballs can be drawn from the bag. The baseballs are replaced after each draw, so each time a baseball is drawn from the bag it could be numbered 1, 2, or 3. Since three draws are made and there are three possible outcomes per draw, there are $3 \times 3 \times 3 = 27$ possible ways to draw three baseballs from the bag.

We provide two solutions to this problem. In Solution 1, we take a direct approach to counting the number of ways a sum of less than 8 can be obtained. In Solution 2, our approach is indirect. We count the number of ways a sum of 8 or more can be obtained, and subtract this number from 27 to obtain the desired sum. In this problem it is actually easier to count the desired sum in this indirect way.

Solution 1

Let's determine how many of the 27 draws result in a sum that is less than 8 by systematically looking at the possible selections.

- Ball 1 is drawn three times. In this case, the sum will be $1 + 1 + 1 = 3 < 8$. This can be done only 1 way: $1, 1, 1$.
- Ball 1 is drawn twice and ball 2 is drawn once. In this case, the sum will be $1 + 1 + 2 = 4 < 8$. This can be done 3 ways: $1, 1, 2$ or $1, 2, 1$ or $2, 1, 1$.
- Ball 1 is drawn twice and ball 3 is drawn once. In this case, the sum will be $1 + 1 + 3 = 5 < 8$. This can be done 3 ways: $1, 1, 3$ or $1, 3, 1$ or $3, 1, 1$.
- Ball 1 is drawn once and ball 2 is drawn twice. In this case, the sum will be $1 + 2 + 2 = 5 < 8$. This can be done 3 ways: $1, 2, 2$ or $2, 1, 2$ or $2, 2, 1$.
- Ball 1 is drawn once and ball 3 is drawn twice. In this case, the sum will be $1 + 3 + 3 = 7 < 8$. This can be done 3 ways: $1, 3, 3$ or $3, 1, 3$ or $3, 3, 1$.

- Ball 1 is drawn once, ball 2 is drawn once, and ball 3 is drawn once. In this case the sum will be $1 + 2 + 3 = 6 < 8$. This can be done 6 ways: $1, 2, 3$ or $1, 3, 2$ or $2, 1, 3$ or $2, 3, 1$ or $3, 1, 2$ or $3, 2, 1$.
- Ball 2 is drawn three times. In this case the sum will be $2 + 2 + 2 = 6 < 8$. This can be done only 1 way: $2, 2, 2$.
- Ball 2 is drawn twice and ball 3 is drawn once. In this case the sum will be $2 + 2 + 3 = 7 < 8$. This can be done 3 ways: $2, 2, 3$ or $2, 3, 2$ or $3, 2, 2$.
- Ball 2 is drawn once and ball 3 is drawn twice. In this case the sum will be $2 + 3 + 3 = 8$, which is not less than 8.
- Ball 3 is drawn three times. In this case the sum will be $3 + 3 + 3 = 9$, which is not less than 8.

We see that there are $1 + 3 + 3 + 3 + 3 + 6 + 1 + 3 = 23$ ways to draw three baseballs so that the sum of the numbers recorded is less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$, or approximately 85%.

Solution 2

Let's determine how many of the 27 draws result in a sum that is 8 or more. Since the maximum sum is 9, we need to count the number of ways the sum can be 8 or 9.

- The sum is 8. The only way to do this is to draw ball 2 once and ball 3 twice. This can be done 3 ways: $2, 3, 3$ or $3, 2, 3$ or $3, 3, 2$.
- The sum is 9. This can be done only 1 way: $3, 3, 3$.

We see that there are $3 + 1 = 4$ ways to draw three baseballs so that the sum of the numbers recorded is 8 or 9. Therefore, of the 27 outcomes, $27 - 4 = 23$ give a sum less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$, or approximately 85%.

The indirect approach used in the second solution is definitely more efficient!

Extension:

Ivy's game is *unfair* since the probability of obtaining a sum less than 8 is $\frac{23}{27}$ or 85% while the probability of obtaining a sum of 8 or higher is $\frac{4}{27}$ or 15%. In a fair game, we want the probability of winning to be the same as the probability of losing. Can you modify Ivy's game to make it fair?