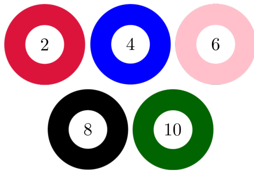

Problem of the Week

Problem D and Solution

No Power

Problem

Five balls are placed in a bag. Each ball is labelled with a 2, 4, 6, 8, or 10, with no ball having the same label as any other. Adeleke randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Then Bo randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Finally, Carlos randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Determine the probability that the product of the three recorded integers is not a power of 2.

Solution

Solution 1

One way to solve this problem is to list out all of the possible choices, calculate the product for each choice, and then count the number of products that are not a power of 2. If we did so, we would find that there are 125 possible choices. Of these, 98 result in a product that is not a power of 2. Therefore, the probability that the product is not a product of 2 is $\frac{98}{125}$. In Solutions 2 and 3, we will see more efficient ways to calculate this probability.

Solution 2

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2. So, to determine the number of choices that result in a product that is not a power of 2, we will count the number of choices that result in a product that is a power of 2, and subtract this from the total number of choices.

Since Adeleke, Bo, and Carlos each have five possible integers they can choose, there are $5 \times 5 \times 5 = 125$ possible choices of integers. For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three chosen integers must be a power of 2. There are three balls labelled with a power of 2, namely, 2, 4, and 8. Therefore, the number of choices that result in a power of 2 is $3 \times 3 \times 3 = 27$.

Since there are 27 choices that give a product that is a power of 2, there must be $125 - 27 = 98$ choices that give a product that is not a power of 2. Therefore, the probability that the product is not a power of 2 is $\frac{98}{125}$.

Solution 3

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2. If p is the probability that the product is a power of 2 and q is the probability that the product is not a power of 2, then $p + q = 1$. Therefore, we can calculate q by calculating p and noting that $q = 1 - p$.

For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three integers must be a power of 2. There are three balls labelled with a power of 2, namely, 2, 4, and 8. Thus, the probability of randomly choosing a ball with a label that is power of 2 is $\frac{3}{5}$. Since Adeleke, Bo, and Carlos choose their integers independently, then the probability that each chooses a power of 2 is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$. In other words, $p = \frac{27}{125}$, and so $q = 1 - p = 1 - \frac{27}{125} = \frac{98}{125}$. Therefore, the probability that the product is not a power of 2 is $\frac{98}{125}$.