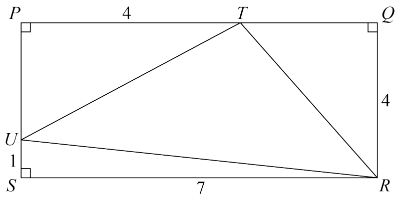


Problem of the Week
Problem D and Solution
Many Ways to Get There

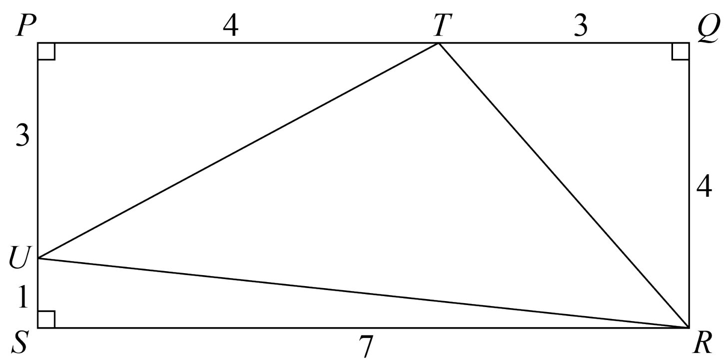
Problem

Rectangle $PQRS$ has $QR = 4$ and $RS = 7$. $\triangle TRU$ is inscribed in rectangle $PQRS$ with T on PQ such that $PT = 4$, and U on PS such that $SU = 1$.

Determine the value of $\angle RUS + \angle PUT$. There are many ways to solve this problem. After you have solved it, see if you can solve it a different way.

Solution

Since $PQRS$ is a rectangle, $PQ = RS$, so $TQ = 3$. Similarly $PS = QR$, so $PU = 3$.



We will now present three different solutions. The first uses the Pythagorean Theorem, the second uses congruent triangles, and the third uses basic trigonometry.

Solution 1

Since $\triangle UPT$ has a right angle at P, we can apply the Pythagorean Theorem to find that $UT^2 = PU^2 + PT^2 = 3^2 + 4^2 = 25$. Therefore, $UT = 5$, since $UT > 0$.

Similarly, since $\triangle TQR$ has a right angle at Q, we can apply the Pythagorean Theorem to find that $TR = 5$.

Since $\triangle RSU$ has a right angle at S, we can apply the Pythagorean Theorem to find that $UR^2 = RS^2 + SU^2 = 7^2 + 1^2 = 50$ and so $UR = \sqrt{50}$, since $UR > 0$.

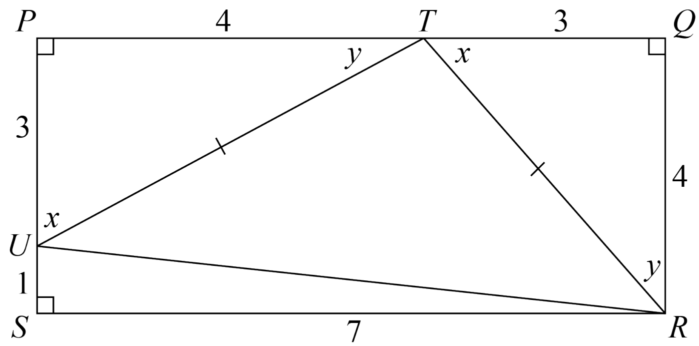
In $\triangle TRU$, notice that $UT^2 + TR^2 = 5^2 + 5^2 = 25 + 25 = 50 = UR^2$. Therefore, $\triangle TRU$ is a right-angled triangle, with $\angle UTR = 90^\circ$. Also, since $UT = TR = 5$, $\triangle TRU$ is an isosceles right-angled triangle, and so $\angle TUR = \angle TRU = 45^\circ$.

The angles in a straight line sum to 180°, so we have $\angle RUS + \angle TUR + \angle PUT = 180^\circ$.

Since $\angle TUR = 45^\circ$, this becomes $\angle RUS + 45^\circ + \angle PUT = 180^\circ$, and so $\angle RUS + \angle PUT = 180^\circ - 45^\circ = 135^\circ$. Therefore, $\angle RUS + \angle PUT = 135^\circ$.

Solution 2

Looking at $\triangle UPT$ and $\triangle TQR$, we have $PT = QR = 4$, $PU = TQ = 3$, and $\angle UPT = \angle TQR = 90^\circ$. Therefore $\triangle UPT \cong \triangle TQR$ by side-angle-side triangle congruency. From the triangle congruency, it follows that $UT = TR$, $\angle QTR = \angle PUT$, and $\angle TRQ = \angle PTU$. Let $\angle QTR = \angle PUT = x$ and $\angle TRQ = \angle PTU = y$.



Since the angles in a triangle sum to 180°, in right-angled $\triangle UPT$, $\angle PUT + \angle PTU = 90^\circ$. That is, $x + y = 90^\circ$.

Since the angles in a straight line sum to 180°, $\angle PTU + \angle UTR + \angle QTR = 180^\circ$. That is, $y + \angle UTR + x = 180^\circ$. Substituting $x + y = 90^\circ$, we obtain $90^\circ + \angle UTR = 180^\circ$, and $\angle UTR = 90^\circ$ follows.

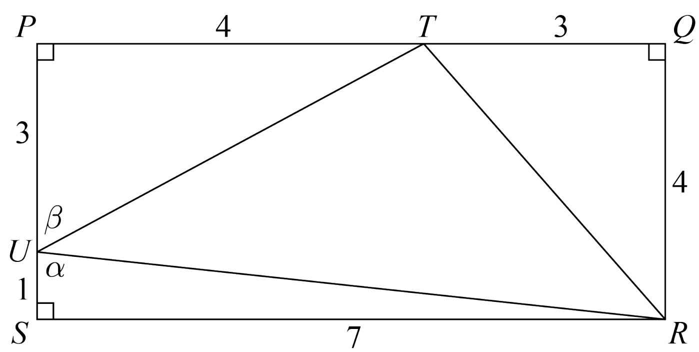
Since $UT = TR$ and $\angle UTR = 90^\circ$, $\triangle TRU$ is an isosceles right-angled triangle and so $\angle TUR = \angle TRU = 45^\circ$.

The angles in a straight line sum to 180°, so we have $\angle RUS + \angle TUR + \angle PUT = 180^\circ$.

Since $\angle TUR = 45^\circ$, this becomes $\angle RUS + 45^\circ + \angle PUT = 180^\circ$, and so $\angle RUS + \angle PUT = 180^\circ - 45^\circ = 135^\circ$. Therefore, $\angle RUS + \angle PUT = 135^\circ$.

Solution 3

Let $\angle RUS = \alpha$ and $\angle PUT = \beta$.



Using basic trigonometry, from right-angled $\triangle RSU$, we have $\tan\alpha = \frac{7}{1} = 7$, and so $\alpha = \tan^{-1}(7)$. Similarly, from right-angled $\triangle UPT$, we have $\tan\beta = \frac{4}{3}$, and so $\beta = \tan^{-1}\left(\frac{4}{3}\right)$.

Then $\angle RUS + \angle PUT = \alpha + \beta = \tan^{-1}(7) + \tan^{-1}\left(\frac{4}{3}\right) = 135^\circ$.

Therefore, $\angle RUS + \angle PUT = 135^\circ$.

This third solution is very efficient and concise. However, some of the beauty is lost as a result of this direct approach.