Problem of the Week
Problem D and Solution
Digits Multiplied

Problem

The digits of any positive integer can be multiplied together to give the *digit product* for the integer. For example, 345 has the digit product of $3 \times 4 \times 5 = 60$. There are many other positive integers that have 60 as a digit product. For example, 2532 and 14 153 both have a digit product of 60. Note that 256 is the smallest positive integer with a digit product of 60.

There are also many positive integers that have a digit product of 2160. Determine the smallest such integer.

Solution

Let N be the smallest positive integer whose digit product is 2160.

In order to find N, we must find the minimum possible number of digits whose product is 2160. This is because if the integer a has more digits than the integer b, then $a > b$. Once we have determined the digits that form N, then the integer N is formed by writing those digits in increasing order.

Note that the digits of N cannot include 0, or else the digit product of N would be 0.

Also, the digits of N cannot include 1, otherwise we could remove the 1 and obtain an integer with fewer digits (and thus, a smaller integer) with the same digit product. Therefore, the digits of N will be between 2 and 9, inclusive.

Since digits of N multiply to 2160, we can use the prime factorization of 2160 to help determine the digits of N:

$$2160 = 2^4 \times 3^3 \times 5$$

In order for the digit product of N to have a factor of 5, one of the digits of N must equal 5.

The digit product of N must also have a factor of $3^3 = 27$. We cannot find one digit whose product is 27 but we can find two digits whose product is 27. In particular, $27 = 3 \times 9$. Therefore, N could also have the digits 3 and 9.

Then the remaining digits of N must have a product of $2^4 = 16$. We need to find a combination of the smallest number of digits whose product is 16. We cannot have one digit whose product is 16, but we can have two digits whose product is 16. In particular, $16 = 2 \times 8$ and $16 = 4 \times 4$.

Therefore, it is possible for N to have 5 digits. We have seen that this can happen when the digits of N are 5, 3, 9, 2, 8 or 5, 3, 9, 4, 4.

However, notice that the product of 2 and 3 is 6. Therefore, rather than using the digits 5, 3, 9, 2, 8, we can replace the two digits 2 and 3 with the single digit 6. We now have the digits 6, 5, 8, and 9. The smallest integer using these digits is 5689.

It is possible that we can take a factor of 2 from the 8 and a factor of 3 from the 9 to make another $2 \times 3 = 6$. However, the digits will be now be 5, 6, 6, 4, and 3. This means we will have a five-digit number which is larger than than the four-digit number 5689.

Therefore, the smallest possible integer with a digit product of 2160 is 5689.