Problem of the Week

Problem D and Solution

What's in That Square?

Problem

Fourteen squares are placed in a row forming the grid below. Each square is to be filled with a positive integer, according to the following rules.

1. The product of any four integers in adjacent squares is 120.
2. Integers may appear more than once in the grid.

Four of the squares are already filled with a positive integer, as shown. Determine all possible values of x.

		2			4			x			3		

Solution

In both solutions, let a_1 be the positive integer in the first square, a_2 the positive integer in the second square, a_3 be the positive integer in the third square, a_4 the positive integer in the fourth square, and so on.

Solution 1

Consider squares 3 to 6. Since the product of any four adjacent integers is 120, we have $2 \times a_4 \times a_5 \times 4 = 120$. Therefore, $a_4 \times a_5 = \frac{120}{2 \times 4} = 15$. Since a_4 and a_5 are positive integers, there are four possibilities: $a_4 = 1$ and $a_5 = 15$, or $a_4 = 15$ and $a_5 = 1$, or $a_4 = 3$ and $a_5 = 5$, or $a_4 = 5$ and $a_5 = 3$.

In each of the four cases, we will have $a_7 = 2$. We can see why by considering squares 4 to 7. We have $a_4 \times a_5 \times 4 \times a_7 = 120$, or $15 \times 4 \times a_7 = 120$, since $a_4 \times a_5 = 15$. Therefore, $a_7 = \frac{120}{15 \times 4} = 2$.

- Case 1: $a_4 = 1$ and $a_5 = 15$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $15 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{15 \times 4 \times 2} = 1$.
  Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $4 \times 2 \times 1 \times x = 120$, or $x = \frac{120}{4 \times 2} = 15$.
  Let's check that $x = 15$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
  Consider squares 9 to 12. If $x = 15$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{15 \times 3} = \frac{8}{3}$. But a_{10} and a_{11} must both be integers, so is not possible for $a_{10} \times a_{11} = \frac{8}{3}$. Therefore, it must not be possible for $a_4 = 1$ and $a_5 = 15$, and so we find that there is no solution for x in this case.

- Case 2: $a_4 = 15$ and $a_5 = 1$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $1 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{4 \times 2} = 15$.

Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times15} = 1$.
Let's check that $x = 1$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider squares 7 to 10. Since $a_7 = 2$, $a_8 = 15$, and $x = 1$, then $a_{10} = \frac{120}{2\times15\times1} = 4$.
Similarly, $a_{11} = \frac{120}{15\times1\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 1 \times 4 \times 2 \times 3 = 24 \neq 120$.
Therefore, it is not possible for $a_4 = 15$ and $a_5 = 1$. There is no solution for x in this case.

- Case 3: $a_4 = 3$ and $a_5 = 5$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $5 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{5\times4\times2} = 3$.
  Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times3} = 5$.
  Let's check that $x = 5$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
  Consider squares 7 to 10. Since $a_7 = 2$, $a_8 = 3$, and $x = 5$, then $a_{10} = \frac{120}{2\times3\times5} = 4$.
  Similarly, $a_{11} = \frac{120}{3\times5\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 5 \times 4 \times 2 \times a_{12} = 120$, so $a_{12} = \frac{120}{5\times4\times2} = 3$. Therefore, the condition that $a_{12} = 3$ is satisfied in the case where $a_4 = 3$ and $a_5 = 5$. If we continue to fill out the entries in the squares, we obtain the entries shown in the diagram below.

5	4	2	3	5	4	2	3	5	4	2	3	5	4

  We see that $x = 5$ is a possible solution. However, is it the only solution? We have one final case to check.

- Case 4: $a_4 = 5$ and $a_5 = 3$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $3 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{3\times4\times2} = 5$.
  Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times5} = 3$.
  Let's check that $x = 3$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
  Consider squares 9 to 12. If $x = 3$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{3\times3} = \frac{40}{3}$. But a_{10} and a_{11} must both be integers, so it is not possible for $a_{10} \times a_{11} = \frac{40}{3}$. Therefore, it must not be possible for $a_4 = 5$ and $a_5 = 3$, and so we find that there is no solution for x in this case.

Therefore, the only possible value for x is $x = 5$.

Solution 2

You may have noticed a pattern for the a_i's in Solution 1. We will explore this pattern.

Since the product of any four adjacent integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$. Since both sides are divisible by $a_2a_3a_4$, and each is a positive integer, then $a_1 = a_5$.
Similarly, $a_2a_3a_4a_5 = a_3a_4a_5a_6 = 120$, and so $a_2 = a_6$.
In general, $a_na_{n+1}a_{n+2}a_{n+3} = a_{n+1}a_{n+2}a_{n+3}a_{n+4}$, and so $a_n = a_{n+4}$.
We can use this along with the given information to fill out the entries in the squares as follows:

x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $4 \times 2 \times 3 \times x = 120$ and so $x = \frac{120}{4\times2\times3} = 5$.