Problem of the Week
Problem E and Solution
Keeping it Small

Problem

Clara is learning to code, so she has written a program to help her practice what she has learned so far.

Clara's program takes two real numbers as input, called A and B. First, her program doubles A, squares the result, and then reduces this result by four times A. The result is called C. Then her program squares B, and then increases this result by six times B. The result is called D. Finally, her program outputs the sum of C and D.

Determine the minimum possible output of Clara's program, and the two input values that produce this output.

Solution

In order to minimize the final output, we need to minimize both C and D.

First, let's minimize C. Clara's program doubles A to get $2A$. It squares this result to get $(2A)^2 = 4A^2$. It then reduces this number by $4A$, to get $C = 4A^2 - 4A$. Thus, we need to minimize $4A^2 - 4A$. This is a quadratic and so represents a parabola. Since the coefficient of A^2 is positive, it opens up and so its minimum value occurs at its vertex. We can find the vertex by completing the square.

$$\begin{aligned} 4A^2 - 4A &= 4(A^2 - A) \\ &= 4\left(A^2 - A + \frac{1}{4} - \frac{1}{4}\right) \\ &= 4\left(A^2 - A + \frac{1}{4}\right) - 1 \\ &= 4\left(A - \frac{1}{2}\right)^2 - 1 \end{aligned}$$

The vertex is at $\left(\frac{1}{2}, -1\right)$, and so the minimum value of $C = 4A^2 - 4A$ is -1 and occurs when $A = \frac{1}{2}$.

Now let's minimize D. Clara's program squares B to get B^2. It then increases the result by $6B$ to get $D = B^2 + 6B$. So we need to minimize $B^2 + 6B$. This is a quadratic and so represents a parabola. Since the coefficient of B^2 is positive, it opens up and so its minimum value occurs at its vertex. We can find the vertex by completing the square.

$$\begin{aligned} B^2 + 6B &= (B^2 + 6B + 9) - 9 \\ &= (B + 3)^2 - 9 \end{aligned}$$

The vertex is at $(-3, -9)$, and so the minimum value of $D = B^2 + 6B$ is -9 and occurs when $B = -3$.

Therefore, the minimum possible output of Clara's program is $-1 + (-9) = -10$ and occurs when $A = \frac{1}{2}$ and $B = -3$.

Aside: This problem is essentially asking us to minimize the multivariable function $f(x, y) = 4x^2 - 4x + y^2 + 6y$.