# Problem of the Week

## Problem E and Solution

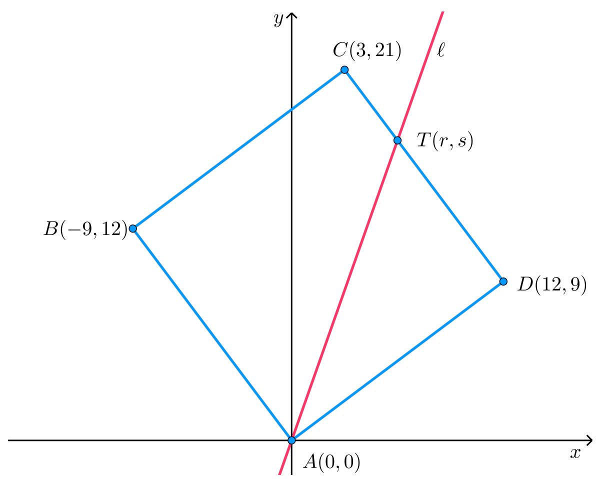
## A Dividing Point

### Problem

A square has vertices at $A(0,0)$, $B(-9,12)$, $C(3,21)$, and $D(12,9)$.

The line $\ell$ passes through $A$ and intersects $CD$ at point $T(r,s)$, splitting the square so that the area of square $ABCD$ is three times the area of $\triangle ATD$.

Determine the equation of line $\ell$.




## Solution

Since $A$ has coordinates $(0,0)$ and $D$ has coordinates $(12,9)$, using the distance formula, we have

$$\begin{aligned} AD &= \sqrt{(9-0)^2 + (12-0)^2} \\ &= \sqrt{81+144} \\ &= \sqrt{225} \\ &= 15 \end{aligned}$$

Therefore, the area of square $ABCD$ is equal to $15^2 = 225$.

Since the area of square $ABCD$ is three times the area of $\triangle ATD$, the area of $\triangle ATD$ is equal to $\frac{1}{3}$ of the area of square $ABCD$. Thus, the area of $\triangle ATD = \frac{1}{3}(225) = 75$.

Since $ABCD$ is a square, $\angle ADC = 90°$. Consider $\triangle ATD$. This triangle is a right-angled triangle with base $AD = 15$ and height $TD$.

Using the formula area $= \dfrac{\text{base} \times \text{height}}{2}$,

$$\begin{aligned} \text{area of } \triangle ATD &= \frac{AD \times TD}{2} \\ 75 &= \frac{15 \times TD}{2} \\ TD &= 10 \end{aligned}$$

From here we present two solutions.

## Solution 1

We first calculate the equation of the line that the segment $CD$ lies on.

Since $D$ has coordinates $(12, 9)$ and $C$ has coordinates $(3, 21)$, this line has slope equal to $\frac{21-9}{3-12} = \frac{12}{-9} = -\frac{4}{3}$.

Since the line has slope $-\frac{4}{3}$ and the point $(3, 21)$ lies on the line, we have

$$\begin{aligned} \frac{y-21}{x-3} &= -\frac{4}{3} \\ 3y - 63 &= -4x + 12 \\ 3y &= -4x + 75 \\ y &= -\frac{4}{3}x + 25 \end{aligned}$$

Since $T(r, s)$ lies on this line, $s = -\dfrac{4}{3}r + 25$.

Using the distance formula, since $T$ has coordinates $(r, s)$, $D$ has coordinates $(12, 9)$, and $TD = 10$, we have

$$\begin{aligned} \sqrt{(r-12)^2 + (s-9)^2} &= 10 \\ (r-12)^2 + (s-9)^2 &= 100 \end{aligned}$$

Since $s = -\frac{4}{3}r + 25$, we have

$$\begin{aligned} (r-12)^2 + \left(\left(-\frac{4}{3}r + 25\right) - 9\right)^2 &= 100 \\ (r-12)^2 + \left(-\frac{4}{3}r + 16\right)^2 &= 100 \\ r^2 - 24r + 144 + \frac{16}{9}r^2 - \frac{128}{3}r + 256 &= 100 \\ \frac{25}{9}r^2 - \frac{200}{3}r + 300 &= 0 \\ \frac{25}{9}(r^2 - 24r + 108) &= 0 \\ r^2 - 24r + 108 &= 0 \\ (r-6)(r-18) &= 0 \\ r &= 6,\ 18 \end{aligned}$$

But $r = 18$ lies outside the square. Therefore, $r = 6$ and $s = -\frac{4}{3}(6) + 25 = -8 + 25 = 17$.

Thus, the line $\ell$ passes through $A(0, 0)$ and $T(6, 17)$, has $y$-intercept 0, and slope $\dfrac{17-0}{6-0} = \dfrac{17}{6}$.

Therefore, the equation of line $\ell$ is $y = \dfrac{17}{6}x$, or $17x - 6y = 0$.

## Solution 2

Since $TD = 10$ and $CD = 15$, we have $CT = CD - TD = 15 - 10 = 5$.

Since $\triangle ATD$ is a right-angled triangle, using the Pythagorean Theorem we have

$$\begin{aligned} AT^2 &= AD^2 + TD^2 \\ (r-0)^2 + (s-0)^2 &= 15^2 + 10^2 \\ r^2 + s^2 &= 325 \end{aligned}$$

Since $T$ has coordinates $(r, s)$, $C$ has coordinates $(3, 21)$, and $CT = 5$, using the distance formula we have

$$\begin{aligned} 5 &= \sqrt{(r-3)^2 + (s-21)^2} \\ 25 &= r^2 - 6r + 9 + s^2 - 42s + 441 \\ 6r + 42s &= r^2 + s^2 + 425 \end{aligned}$$

Since $r^2 + s^2 = 325$, we have

$$\begin{aligned} 6r + 42s &= 325 + 425 \\ &= 750 \end{aligned}$$

Again, since $T$ has coordinates $(r, s)$, $D$ has coordinates $(12, 9)$, and $TD = 10$, using the distance formula we have

$$\begin{aligned} 10 &= \sqrt{(r-12)^2 + (s-9)^2} \\ 100 &= r^2 - 24r + 144 + s^2 - 18s + 81 \\ 24r + 18s &= r^2 + s^2 + 125 \end{aligned}$$

Since $r^2 + s^2 = 325$, we have

$$\begin{aligned} 24r + 18s &= 325 + 125 \\ &= 450 \end{aligned}$$

We now have the system of equations

$$\begin{aligned} 6r + 42s &= 750 \\ 24r + 18s &= 450 \end{aligned}$$

Multiplying the first equation by 4, we get the system

$$\begin{aligned} 24r + 168s &= 3000 \\ 24r + 18s &= 450 \end{aligned}$$

Subtracting the second equation from the first gives $150s = 2550$, and $s = 17$ follows. Substituting $s = 17$ into $6r + 42s = 750$, we obtain $6r + 42(17) = 750$, and $r = 6$ follows.

Thus, the line $\ell$ passes through $A(0, 0)$ and $T(6, 17)$, has $y$-intercept 0, and slope $\dfrac{17-0}{6-0} = \dfrac{17}{6}$.

Therefore, the equation of line $\ell$ is $y = \dfrac{17}{6}x$, or $17x - 6y = 0$.

**Extension:**

Can you determine the coordinates of point $U$ on $CB$ such that the area of $\triangle ABU$ is equal to the area of $\triangle ATD$? By finding $U$ and $T$, you will have found two line segments, $AU$ and $AT$, that divide square $ABCD$ into three regions of equal area.