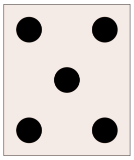

Problem of the Week

Problem E and Solution

Dot Remover

Problem

Manfred has six cards. One card has 2 dots on it, one card has 3 dots on it, one card has 4 dots on it, one card has 5 dots on it, one card has 6 dots on it, and one card has 7 dots on it.

Manfred removes one of the dots at random, with each of the 27 dots equally likely to be removed. Esther then randomly chooses one of the cards, with each card equally likely to be chosen.

What is the probability that the card chosen by Esther has an odd number of dots on it?

Solution

Initially, there are three cards with an even number of dots and three cards with an odd number of dots. When a dot is removed from a card with an even number of dots, that card then has an odd number of dots. When a dot is removed from a card with an odd number of dots, that card then has an even number of dots.

If a dot is removed from a card with an even number of dots, then there will be four cards with an odd number of dots and two cards with an even number of dots. This means that the probability of choosing a card with an odd number of dots after a dot is removed is $\frac{4}{6} = \frac{2}{3}$ in this case.

If a dot is removed from a card with an odd number of dots, then there will be two cards with an odd number of dots and four cards with an even number of dots. This means that the probability of choosing a card with an odd number of dots after a dot is removed is $\frac{2}{6} = \frac{1}{3}$ in this case.

Since there are $2+3+4+5+6+7 = 27$ dots in total, then the probability that a dot is removed from the card with 2 dots is $\frac{2}{27}$, from the card with 3 dots is $\frac{3}{27}$, and so on. Thus, the probability that a dot is removed from the card with 2 dots and then a card with an odd number of dots is chosen is $\frac{2}{27} \times \frac{2}{3}$, since there are now four cards with an odd number of dots and two cards with an even number of dots.

Similarly, the probability that a dot is removed from the card with 3 dots and then a card with odd number of dots is chosen is $\frac{3}{27} \times \frac{1}{3}$.

Continuing in this way, the probability of choosing a card with an odd number of dots after a dot is removed is $\frac{2}{27} \times \frac{2}{3} + \frac{3}{27} \times \frac{1}{3} + \frac{4}{27} \times \frac{2}{3} + \frac{5}{27} \times \frac{1}{3} + \frac{6}{27} \times \frac{2}{3} + \frac{7}{27} \times \frac{1}{3}$.

This is equal to

$\frac{2}{3}\left(\frac{2}{27} + \frac{4}{27} + \frac{6}{27}\right) + \frac{1}{3}\left(\frac{3}{27} + \frac{5}{27} + \frac{7}{27}\right) = \frac{2}{3}\left(\frac{12}{27}\right) + \frac{1}{3}\left(\frac{15}{27}\right) = \frac{8}{27} + \frac{5}{27} = \frac{13}{27}$.