



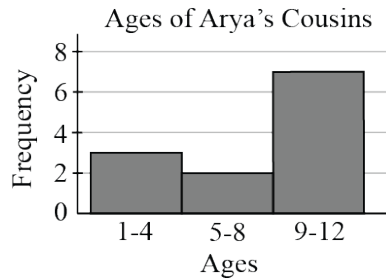
Problem of the Week

Problem E and Solution

Twelve Cousins

Problem

Arya has twelve cousins. She secretly writes their ages, as positive integers, on a piece of paper. She then tells you the mean (average) of the twelve ages is 8.75 and the median is 10. Finally, she draws you the following histogram to represent the ages of her cousins.



How many possibilities are there for the ages of Arya's twelve cousins?

Solution

Since there are 12 ages, the median will be the number halfway between the sixth and seventh ages when they are arranged in increasing order. From the histogram, we know that the sixth and seventh ages are both between 9 and 12. Since the median is 10, it follows that the sixth and seventh ages are either 9 and 11, respectively, or they are both 10.

Since the mean of the 12 ages is 8.75, it follows that the sum of all 12 ages is $8.75 \times 12 = 105$. We know from the previous paragraph that the sixth and seventh ages are either 9 and 11, respectively, or they are both 10. Either way, their sum is 20. So the sum of the remaining ten ages is $105 - 20 = 85$. From the histogram, we know that three of these ages are between 1 and 4, two of these ages are between 5 and 8, and the remaining five are between 9 and 12 and also greater than the seventh age, which is either 10 or 11.

We will consider cases by looking at the possible sums of the five smallest ages.

Case 1: The sum of the five smallest ages is 28.

This is the maximum possible sum of the five smallest ages, and occurs when the ages are 4, 4, 4, 8, 8. The corresponding sum of the five largest ages is then $85 - 28 = 57$.

- If the seventh age is 10, then the five largest ages could be 10, 11, 12, 12, 12 or 11, 11, 11, 12, 12. Thus, there are 2 possibilities for the ages of Arya's cousins.
- If the seventh age is 11, then the five largest ages could only be 11, 11, 11, 12, 12. Thus, there is 1 possibility for the ages of Arya's cousins.

In total, there are $2 + 1 = 3$ possibilities for the ages of Arya's cousins if the sum of the five smallest ages is 28.



Case 2: The sum of the five smallest ages is 27.

This occurs when the ages are 3, 4, 4, 8, 8 or 4, 4, 4, 7, 8. The corresponding sum of the five largest ages is then $85 - 27 = 58$.

- If the seventh age is 10, then the five largest ages could be 10, 12, 12, 12, 12 or 11, 11, 12, 12, 12. Thus, there are 2 possibilities for the five smallest ages and 2 possibilities for the five largest ages, which results in $2 \times 2 = 4$ possibilities for the ages of Arya's cousins.
- If the seventh age is 11, then the five largest ages could only be 11, 11, 12, 12, 12. Thus, there are 2 possibilities for the five smallest ages and 1 possibility for the five largest ages, which results in $2 \times 1 = 2$ possibilities for the ages of Arya's cousins.

In total, there are $4 + 2 = 6$ possibilities for the ages of Arya's cousins if the sum of the five smallest ages is 27.

Case 3: The sum of the five smallest ages is 26.

This occurs when the ages are 2, 4, 4, 8, 8 or 3, 3, 4, 8, 8 or 3, 4, 4, 7, 8 or 4, 4, 4, 6, 8 or 4, 4, 4, 7, 7. The corresponding sum of the five largest ages is then $85 - 26 = 59$. The only possibility for the five largest ages is 11, 12, 12, 12, 12. Thus, there are 5 possibilities for the five smallest ages, 2 possibilities for the two middle ages, and 1 possibility for the five largest ages, which results in $5 \times 2 \times 1 = 10$ possibilities for the ages of Arya's cousins if the sum of the five smallest ages is 26.

Case 4: The sum of the five smallest ages is 25.

This occurs when the ages are 1, 4, 4, 8, 8 or 2, 3, 4, 8, 8 or 2, 4, 4, 7, 8 or 3, 3, 3, 8, 8 or 3, 3, 4, 7, 8 or 3, 4, 4, 6, 8 or 3, 4, 4, 7, 7 or 4, 4, 4, 5, 8 or 4, 4, 4, 6, 7. The corresponding sum of the five largest ages is then $85 - 25 = 60$. This is the maximum possible sum of the five largest ages and occurs when the ages are 12, 12, 12, 12, 12. Thus, there are 9 possibilities for the five smallest ages, 2 possibilities for the two middle ages, and 1 possibility for the five largest ages, which results in $9 \times 2 \times 1 = 18$ possibilities for the ages of Arya's cousins if the sum of the five smallest ages is 25.

Therefore, in total, there are $3 + 6 + 10 + 18 = 37$ possibilities for the ages of Arya's twelve cousins.