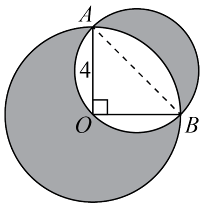


Problem of the Week

Problem E and Solution

Only One

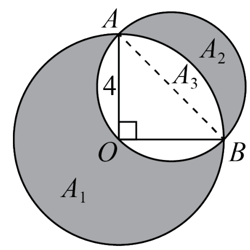
Problem

A circle with centre O and radius 4 has points A and B on its circumference such that $\angle AOB = 90^\circ$.

Another circle with diameter AB is drawn such that O lies on its circumference.

Find the area of the shaded region, which is the area inside one circle or the other circle, but not both.

Solution



Let A_1 be the region inside the larger circle but outside the smaller circle.
Let A_2 be the region inside the smaller circle but outside the larger circle.
Let A_3 be the region inside sector AOB but outside of $\triangle AOB$.
We need to calculate $A_1 + A_2$.

First, we will calculate A_3.
Since $\angle AOB = 90^\circ$, the area of sector AOB is $\frac{90}{360} = \frac{1}{4}$ the area of the larger circle.
That is, the area of sector AOB is $\frac{1}{4} \times \pi(4)^2 = 4\pi$.
The area of $\triangle AOB$ is $\frac{1}{2}(OA)(OB) = \frac{1}{2}(4)(4) = 8$.
Therefore, $A_3 =$ area of sector $AOB -$ area of $\triangle AOB = (4\pi - 8)$.

Next we will calculate A_2.
Since $\angle AOB = 90^\circ$, the Pythagorean theorem tells us $AB^2 = OA^2 + OB^2 = 4^2 + 4^2 = 32$.
Therefore, $AB = \sqrt{32} = 4\sqrt{2}$, since $AB > 0$.

Since AB is a diameter of the smaller circle, the radius is $\frac{1}{2}AB = \frac{1}{2}\left(4\sqrt{2}\right) = 2\sqrt{2}$.
Therefore, $A_2 + A_3 = \frac{1}{2}$(the area of the circle with radius $2\sqrt{2}$) $= \frac{1}{2}\pi(2\sqrt{2})^2 = \frac{1}{2}\pi(8) = 4\pi$.
Therefore, $A_2 = 4\pi - A_3 = 4\pi - (4\pi - 8) = 8$.

Finally, we will calculate A_1.
A_1 represents the area inside the larger circle which is not in the smaller circle.
The larger circle has radius 4 and area $\pi(4)^2 = 16\pi$.
The smaller circle has radius $2\sqrt{2}$ and area $\pi(2\sqrt{2})^2 = 8\pi$.

$$\begin{aligned}
A_1 &= (\text{area of larger circle}) - \frac{1}{2}(\text{area of smaller circle}) - A_3 \\
&= 16\pi - \frac{1}{2}(8\pi) - (4\pi - 8) \\
&= 16\pi - 4\pi - 4\pi + 8 \\
&= 8\pi + 8
\end{aligned}$$

Therefore, the area of the shaded region is equal to $A_1 + A_2 = (8\pi + 8) + 8 = (8\pi + 16)$ units^2.