Problem of the Week
Problem E and Solution
A Rectangle and a Square

Problem

Simeon has a rope that is 108 cm long and is asked to cut the rope once so that one of the pieces can be arranged, with its two ends touching, to form a square, and the other piece can be arranged, with its two ends touching, to form a rectangle with one side length of 6 cm. Furthermore, the area of the square will be equal to the area of the rectangle.

Where should Simeon make the cut to the original piece of rope?

Solution

Let the length of the piece of rope used to form the square be $4x$ cm. This is also equal to the perimeter of the square. Then the side length of the square is $4x \div 4 = x$ cm. The area of the square is

$$x \times x = x^2 \text{ cm}^2 \tag{1}$$

The length of the piece of rope used to form the rectangle is $(108 - 4x)$ cm. This is also equal to the perimeter of the rectangle. If one side length of the rectangle is 6 cm, then there is $108 - 4x - 6 - 6 = (96 - 4x)$ cm left to form the lengths of the two other sides of the rectangle. Therefore, the other side length of the rectangle is $\frac{96-4x}{2} = (48 - 2x)$ cm. Thus, the area of the rectangle is

$$(6)(48 - 2x) = (288 - 12x) \text{ cm}^2 \tag{2}$$

We are given that the area of the square is equal to the area of the rectangle. So, by equating equations (1) and (2), we obtain

$$x^2 = 288 - 12x$$
$$x^2 + 12x - 228 = 0$$
$$(x - 12)(x + 24) = 0$$

Thus, $x = 12$ or $x = -24$. Since x is the length of the side of the square, we must have $x > 0$. Therefore, $x = 12$ cm. Then the length of rope used to form the square is $4x = 4(12) = 48$ cm.

Therefore, the cut should be made 48 cm from one end (and so 60 cm from the other end), creating a 60 cm piece for the rectangle and a 48 cm piece for the square.

NOTE:
The area of the square is $12 \times 12 = 144$ cm^2.
The length of the other side of the rectangle is $48 - 2x = 48 - 24 = 24$ cm. The area of the rectangle is $24 \times 6 = 144$ cm^2.
(These calculations were not required but are provided as a check of the correctness of the result.)