Problem of the Week

$3, 6, 9, \ldots, 2400$

Problem E and Solution

All Square

Problem

The positive multiples of 3 from 3 to 2400, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.

Solution

What does the prime factorization of a perfect square look like? Let's look at a few examples: $9 = 3^2$, $36 = 6^2 = 2^2 3^2$, and $129\,600 = 360^2 = 2^6 3^4 5^2$. Notice that the exponent on each of the prime factors in the prime factorization in each of the three examples is an even number. In fact, a positive integer is a perfect square exactly when the exponent on each prime in its prime factorization is even. Can you convince yourself that this is true?

The positive integer n is the smallest positive integer such that

$$3n + 6n + 9n + \cdots + 2394n + 2397n + 2400n \tag{1}$$

is a perfect square.

Factoring expression (1), we obtain

$$3n(1 + 2 + 3 + \cdots + 798 + 799 + 800)$$

Then, using the formula $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$, with $n = 800$, we see that this expression is equal to

$$3n\left(\frac{800 \times 801}{2}\right) = 3n(400)(801)$$

Factoring $3 \times 400 \times 801$ into the product of primes, we have that expression (1) is equal to

$$3n[(2)(2)(2)(2)(5)(5)][(3)(3)(89)] = n(2^4)(5^2)(3^3)(89) \tag{2}$$

We need to determine what additional factors are required to make the quantity in expression (2) a perfect square such that n is as small as possible. In order for the exponent on each prime in the prime factorization to be even, we need n to be $3 \times 89 = 267$. Then the quantity in expression (2) is the perfect square

$$n(2^4)(5^2)(3^3)(89) = (3)(89)(2^4)(5^2)(3^3)(89) = (2^4)(5^2)(3^4)(89^2) = \left[(2^2)(5)(3^2)(89)\right]^2$$

Therefore, the smallest positive integer is 267 and the perfect square is

$$267 \times 3 \times 400 \times 801 = 256\,640\,400 = (16\,020)^2$$

NOTE: When solving this problem, we could have instead noticed that
$3n + 6n + 9n + \cdots + 2400n$ is an arithmetic series with $t_1 = 3n$ and $t_{800} = 2400n$.

Substituting these values for t_1 and t_{800} into the formula for the sum of the terms in an arithmetic series, we get

$$S_{800} = \frac{800}{2}(3n + 2400n) = 400(2403n)$$

When we factor $400(2403n)$ into the product of primes, we get the same expression as (2), and then we can continue from there to get the solution of 267.