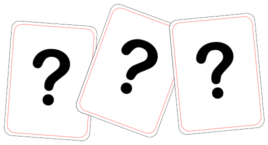

Problem of the Week
Problem E and Solution
Secret Numbers

Problem

Wakana, Yousef, and Zora are each given a card with a positive integer on it. They cannot see each other's cards, but are told that the sum of their three numbers is 14. They then make the following observations.

- Wakana says "I know that Yousef and Zora have different numbers."
- Yousef then says "I already knew that all three numbers were different."
- Zora then says "I now know what all three of the numbers are."

What number does each person have?

Solution

We want to find the single solution to the problem $x + y + z = 14$ that satisfies the observations made by Wakana, Yousef, and Zora. It turns out that there are 78 different possible sums of three positive integers totalling 14. We could list all of the possible solutions and then proceed through the observations until we determine the required solution. However, our approach will be far less exhausting. At the end of the solution, we will provide a justification as to why there are 78 positive integer solutions to the equation $x + y + z = 14$.

The sum of the three numbers is 14, an even number. To generate an even sum, the three numbers must all be even, or one of the numbers must be even and the other two numbers must be odd. We will go through each of the three observations to determine the three numbers.

- First, Wakana says "I know that Yousef and Zora have different numbers."

  How can Wakana KNOW? If her number is even, then Yousef and Zora could both have even numbers or both have odd numbers to generate the sum 14. So if her number was even, Wakana would not KNOW that the other two numbers were different. Therefore, Wakana must have an odd number, one of the others has an odd number and the other has an even number.

- Then, Yousef says "I already knew that all three numbers were different."

  Using the same logic as before, since Yousef knows Wakana and Zora have different numbers, Yousef must have an odd number (and thus Zora must have the even number). But how does Yousef KNOW that all three numbers are different?

  If Yousef has a $1,\ 3$, or 5, Wakana could have the same number, since $1 + 1 + 12$, $3 + 3 + 8$, and $5 + 5 + 4$ all equal 14. So Yousef cannot have a $1,\ 3$, or 5.

  If Yousef has a $7,\ 9,\ 11$, or 13, Wakana could not have the same number as Yousef, in order for the three numbers to sum to 14. Furthermore, if Yousef has a 7, Wakana must have a 5 or lower. If Yousef has a 9, Wakana must have a 3 or lower. If Yousef has an 11, Wakana must have a 1. Yousef cannot have a 13, since the three numbers to sum to 14.

At this point our list of possible solutions has dropped from 78 to 6. The remaining possibilities are shown in the table.

Yousef's Number	Wakana's Number	Zora's Number
7	1	6
7	3	4
7	5	2
9	1	4
9	3	2
11	1	2

- Finally, Zora says "I now know what all three of the numbers are."

  If Zora has a 2, Yousef could have a 7 and Wakana could have a 5, or Yousef could have a 9 and Wakana could have a 3, or Yousef could have an 11 and Wakana could have a 1. So Zora cannot have a 2.

  If Zora has a 4, Yousef could have a 7 and Wakana could have a 3, or Yousef could have a 9 and Wakana could have a 1. So Zora cannot have a 4.

  However, if Zora has a 6, then Yousef must have a 7 and Wakana must have a 1. This is the only possibility in which Zora's statement is true.

  Therefore, Wakana has a 1, Yousef has a 7, and Zora has a 6.

Note: We will show here why there are 78 solutions to the equation $x + y + z = 14$, where x, y, and z are positive integers.

Suppose $x = 1$. Then we have the following possibilities for y and z.

y	1	2	3	4	5	6	7	8	9	10	11	12
z	12	11	10	9	8	7	6	5	4	3	2	1

In total, there are 12 positive integer solutions when $x = 1$.

Suppose $x = 2$. Then we have the following possibilities for y and z.

y	1	2	3	4	5	6	7	8	9	10	11
z	11	10	9	8	7	6	5	4	3	2	1

In total, there are 11 positive integer solutions when $x = 2$.

Continuing in this manner, we can find that there are 10 positive integer solutions when $x = 3$, 9 positive integer solutions when $x = 4$, 8 positive integer solutions when $x = 5$, 7 positive integer solutions when $x = 6$, 6 positive integer solutions when $x = 7$, 5 positive integer solutions when $x = 8$, 4 positive integer solutions when $x = 9$, 3 positive integer solutions when $x = 10$, 2 positive integer solutions when $x = 11$, and 1 positive integer solution when $x = 12$.

In total, there are

$$12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 78$$

positive integer solutions to $x + y + z = 14$.