



Problem of the Week

Problem E and Solution

Fill ALL the Squares

Problem

Twelve squares are placed in a row forming the grid below. Each square is to be filled with an integer. After the third square, each integer in a square is the sum of the previous three integers. If we know the third integer is 6, the sixth integer is 11, and the eleventh integer is 14, determine all of the integers in the grid.

		6			11					14	
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Solution

Let a_1 be the first integer in the grid, a_2 be the second, a_3 be the third, a_4 be the fourth, and so on, until a_{12} which is the twelfth integer in the grid. We are given that $a_3 = 6$, $a_6 = 11$, and $a_{11} = 14$.

Each integer after the third integer is equal to the sum of the previous three integers. Therefore, $a_6 = a_3 + a_4 + a_5$. Thus, $11 = 6 + a_4 + a_5$ or $a_4 + a_5 = 5$.

We also have $a_7 = a_4 + a_5 + a_6 = a_4 + a_5 + 11 = 5 + 11 = 16$, since $a_4 + a_5 = 5$.

Similarly,

$$a_9 = a_6 + a_7 + a_8 = 11 + 16 + a_8 = 27 + a_8,$$

$$a_{10} = a_7 + a_8 + a_9 = 16 + a_8 + a_9 = 16 + (a_8) + (a_8 + 27) = 2a_8 + 43, \text{ and}$$

$$a_{11} = a_8 + a_9 + a_{10} = (a_8) + (a_8 + 27) + (2a_8 + 43) = 4a_8 + 70.$$

We are given that $a_{11} = 14$. Therefore, $4a_8 + 70 = 14$, or $4a_8 = -56$, or $a_8 = -14$.

Therefore, $a_9 = a_8 + 27 = -14 + 27 = 13$, and $a_{10} = 2a_8 + 43 = 2(-14) + 43 = 15$.

Also, $a_{12} = a_9 + a_{10} + a_{11} = 13 + 15 + 14 = 42$.

So far, we know that the integers in the grid, from left to right are

$$a_1, a_2, 6, a_4, a_5, 11, 16, -14, 13, 15, 14, 42$$

Working backwards, $-14 = a_5 + 11 + 16$, so $a_5 = -41$.

From earlier, $a_4 + a_5 = 5$. Since $a_5 = -41$, we know $a_4 = 46$.

Continuing working backwards, $a_5 = a_2 + 6 + a_4$, so $-41 = a_2 + 6 + 46$, or $a_2 = -93$.

Finally, $a_4 = a_1 + a_2 + 6$, so $46 = a_1 + (-93) + 6$, or $a_1 = 133$.

Therefore, the twelve integers in the grid, from left to right, are

$$133, -93, 6, 46, -41, 11, 16, -14, 13, 15, 14, 42$$

The filled grid is shown below. We can indeed check that each integer after the first three integers is equal to the sum of the previous three integers.

133	-93	6	46	-41	11	16	-14	13	15	14	42
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