Problem of the Week
Problem E and Solution
A Lot of Zeros

Problem

For a positive integer n, the product of the integers from 1 to n can be written in abbreviated form as $n!$, which we read as "n *factorial*". So,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

For example,
$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$, and $11! = 11 \times 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 39\,916\,800$.

Note that $6!$ ends in one zero and $11!$ ends in two zeros.

Determine the smallest positive integer n such that $n!$ ends in exactly 1000 zeros.

Solution

When finding a solution to this problem, it may be helpful to work with possible values for n to determine the number of zeros that $n!$ ends in. One could use a calculator as part of this, but many standard calculators switch to scientific notation around $14!$. A trial and error approach could work but it may be very time consuming. Our approach will be more systematic.

A zero is added to the end of a positive integer when we multiply by 10. Multiplying a number by 10 is the same as multiplying a number by 2 and then by 5, or by 5 and then by 2, since $2 \times 5 = 10$ and $5 \times 2 = 10$.

So we want n to be the smallest positive integer such that the prime factorization of $n!$ contains 1000 5s and 1000 2s. Every even positive integer has a 2 in its prime factorization and every positive integer that is a multiple of 5 has a 5 in its prime factorization. There are more positive integers less than or equal to n that are multiples of 2 than multiples of 5. So once we find a positive integer n such that $n!$ has 1000 5s in its prime factorization, we can stop, we know that there will be a sufficient number of 2s in its prime factorization.

There are $\left\lfloor \frac{n}{5} \right\rfloor$ positive integers less than or equal to n that are divisible by 5. Note, the notation $\lfloor x \rfloor$ means *the floor of* x and is the largest integer less than or equal to x. So $\lfloor 4.2 \rfloor = 4$, $\lfloor 4.9 \rfloor = 4$ and $\lfloor 4 \rfloor = 4$. Also, since $5 \times 1000 = 5000$, we know that $n \leq 5000$.

Numbers that are divisible by 25 will add an additional factor of 5, since $25 = 5 \times 5$.
There are $\left\lfloor \frac{n}{25} \right\rfloor$ positive integers less than or equal to n that are divisible by 25.

Numbers that are divisible by 125 will add an additional factor of 5, since $125 = 5 \times 5 \times 5$ and two of the factors have already been counted when we looked at 5 and 25.
There are $\left\lfloor \frac{n}{125} \right\rfloor$ positive integers less than or equal to n that are divisible by 125.

Numbers that are divisible by 625 will add an additional factor of 5, since $625 = 5 \times 5 \times 5 \times 5$ and three of the factors have already been counted when we looked at 5, 25, and 125.
There are $\left\lfloor \frac{n}{625} \right\rfloor$ positive integers less than or equal to n that are divisible by 625.

Numbers that are divisible by 3125 will add an additional factor of 5, since $3125 = 5^5$ and four of the factors have already been counted when we looked at 5, 25, 125, and 625.

There are $\left\lfloor \frac{n}{3125} \right\rfloor$ positive integers less than or equal to n that are divisible by 3125.

The next power of 5 to consider is $5^6 = 15\,625$. But since $n \leq 5000$, we do not need to consider this power of 5 or any larger power.

Thus, we know that n must satisfy the equation

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \left\lfloor \frac{n}{625} \right\rfloor + \left\lfloor \frac{n}{3125} \right\rfloor = 1000$$

Let's ignore the floor function. We know that n is going to be *close* to satisfying

$$\begin{aligned} \frac{n}{5} + \frac{n}{25} + \frac{n}{125} + \frac{n}{625} + \frac{n}{3125} &= 1000 \\ \frac{625n}{3125} + \frac{125n}{3125} + \frac{25n}{3125} + \frac{5n}{3125} + \frac{n}{3125} &= 1000 \\ \frac{781}{3125}n &= 1000 \\ n &= \frac{1000 \times 3125}{781} \\ n &\approx 4001.2 \end{aligned}$$

The number of zeros at the end of 4001! is equal to

$$\begin{aligned} &\left\lfloor \frac{4001}{5} \right\rfloor + \left\lfloor \frac{4001}{25} \right\rfloor + \left\lfloor \frac{4001}{125} \right\rfloor + \left\lfloor \frac{4001}{625} \right\rfloor + \left\lfloor \frac{4001}{3125} \right\rfloor \\ &= \lfloor 800.2 \rfloor + \lfloor 160.04 \rfloor + \lfloor 32.008 \rfloor + \lfloor 6.4016 \rfloor + \lfloor 1.28032 \rfloor \\ &= 800 + 160 + 32 + 6 + 1 \\ &= 999 \end{aligned}$$

Therefore, the number 4001! ends in 999 zeros. We need one more factor of 5 in order to have 1000 zeros at the end. The first integer after 4001 that is divisible by 5 is 4005.

Therefore, 4005 is the smallest positive integer such that 4005! ends in 1000 zeros.

Indeed, we can check. The number of zeros at the end of 4004! is equal to the number of 5s in its prime factorization, which is equal to

$$\begin{aligned} &\left\lfloor \frac{4004}{5} \right\rfloor + \left\lfloor \frac{4004}{25} \right\rfloor + \left\lfloor \frac{4004}{125} \right\rfloor + \left\lfloor \frac{4004}{625} \right\rfloor + \left\lfloor \frac{4004}{3125} \right\rfloor \\ &= \lfloor 800.8 \rfloor + \lfloor 160.16 \rfloor + \lfloor 32.032 \rfloor + \lfloor 6.4064 \rfloor + \lfloor 1.28128 \rfloor \\ &= 800 + 160 + 32 + 6 + 1 \\ &= 999 \end{aligned}$$

The number of zeros at the end of 4005! is equal to the number of 5s in its prime factorization, which is equal to

$$\begin{aligned} &\left\lfloor \frac{4005}{5} \right\rfloor + \left\lfloor \frac{4005}{25} \right\rfloor + \left\lfloor \frac{4005}{125} \right\rfloor + \left\lfloor \frac{4005}{625} \right\rfloor + \left\lfloor \frac{4005}{3125} \right\rfloor \\ &= \lfloor 801 \rfloor + \lfloor 160.2 \rfloor + \lfloor 32.04 \rfloor + \lfloor 6.408 \rfloor + \lfloor 1.2816 \rfloor \\ &= 801 + 160 + 32 + 6 + 1 \\ &= 1000 \end{aligned}$$