Problem of the Week
Problem D and Solution
The Same Power

Problem
Sometimes two powers that are not written with the same base are still equal in value. For example, \(9^3 = 27^2\) and \((-5)^4 = 25^2\).

If \(x\) and \(y\) are integers, find all ordered pairs \((x, y)\) that satisfy the equation

\[(x - 1)^{x+y} = 8^2\]

Solution
Since \(8^2 = 64\), we want to look at how we can express 64 as \(a^b\) where \(a\) and \(b\) are integers. There are six ways to do so. We can do so as \(64^1, 8^2, 4^3, 2^6, (-2)^6,\) and \((-8)^2\).

We use these powers and the expression \((x - 1)^{x+y}\) to find values for \(x\) and \(y\).

- The power \((x - 1)^{x+y}\) is expressed as \(64^1\) when \(x - 1 = 64\) and \(x + y = 1\). Then \(x = 65\) and \(y = -64\) follows. Thus \((65, -64)\) is one pair.
- The power \((x - 1)^{x+y}\) is expressed as \(8^2\) when \(x - 1 = 8\) and \(x + y = 2\). Then \(x = 9\) and \(y = -7\) follows. Thus \((9, -7)\) is one pair.
- The power \((x - 1)^{x+y}\) is expressed as \(4^3\) when \(x - 1 = 4\) and \(x + y = 3\). Then \(x = 5\) and \(y = -2\) follows. Thus \((5, -2)\) is one pair.
- The power \((x - 1)^{x+y}\) is expressed as \(2^6\) when \(x - 1 = 2\) and \(x + y = 6\). Then \(x = 3\) and \(y = 3\) follows. Thus \((3, 3)\) is one pair.
- The power \((x - 1)^{x+y}\) is expressed as \((-2)^6\) when \(x - 1 = -2\) and \(x + y = 6\). Then \(x = -1\) and \(y = 7\) follows. Thus \((-1, 7)\) is one pair.
- The power \((x - 1)^{x+y}\) is expressed as \((-8)^2\) when \(x - 1 = -8\) and \(x + y = 2\). Then \(x = -7\) and \(y = 9\) follows. Thus \((-7, 9)\) is one pair.

Therefore, there are six ordered pairs that satisfy the equation. They are \((65, -64), (9, -7), (5, -2), (3, 3), (-1, 7),\) and \((-7, 9)\).