Problem of the Week
Problem D and Solution
Pi Squares

Problem
Pi Day is an annual celebration of the mathematical constant $\pi$. Pi Day is observed on March 14, since 3, 1, and 4 are the first three significant digits of $\pi$.

Archimedes determined lower bounds for $\pi$ by finding the perimeters of regular polygons inscribed in a circle with diameter of length 1. (An inscribed polygon of a circle has all of its vertices on the circle.) He also determined upper bounds for $\pi$ by finding the perimeters of regular polygons circumscribed in a circle with diameter of length 1. (A circumscribed polygon of a circle has all sides tangent to the circle. That is, each side of the polygon touches the circle in one spot.)

In this problem, we will determine a lower bound for $\pi$ and an upper bound for $\pi$ by considering an inscribed square and a circumscribed square in a circle of diameter 1.

Consider a circle with centre $C$ and diameter 1. Since the circle has diameter 1, it has circumference equal to $\pi$. Now consider the inscribed square $ABDE$ and the circumscribed square $FGHJ$.

The perimeter of square $ABDE$ will be less than the circumference of the circle, $\pi$, and will thus give us a lower bound for the value of $\pi$. The perimeter of square $FGHJ$ will be greater than the circumference of the circle, $\pi$, and will thus give us an upper bound for the value of $\pi$.

Using these squares, determine a lower bound and an upper bound for $\pi$.

Note: For this problem, you may want to use the following known results about circles:

1. For a circle with centre $C$, the diagonals of an inscribed square meet at 90° at $C$.
2. For a circle with centre $C$, the diagonals of a circumscribed square meet at 90° at $C$.
3. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.
Solution

For the inscribed square $ABDE$, draw line segments $AC$ and $BC$. Both $AC$ and $BC$ are radii of the circle with diameter 1, so $AC = BC = 0.5$.

Since the diagonals of square $ABDE$ meet at $90^\circ$ at $C$, it follows that $\triangle ACB$ is a right-angled triangle with $\angle ACB = 90^\circ$. We can use the Pythagorean Theorem to find the length of $AB$.

\[
AB^2 = AC^2 + BC^2
= (0.5)^2 + (0.5)^2
= 0.25 + 0.25
= 0.5
\]

Therefore, $AB = \sqrt{0.5}$, since $AB > 0$.

Since $AB$ is one of the sides of the inscribed square, the perimeter of square $ABDE$ is equal to $4 \times AB = 4\sqrt{0.5}$. This gives us a lower bound for $\pi$. That is, we know $\pi > 4\sqrt{0.5} \approx 2.828$.

For the circumscribed square, let $M$ be the point of tangency on side $FJ$ and let $N$ be the point of tangency on $GH$. Draw radii $CM$ and $CN$. Since $M$ is a point of tangency, we know that $\angle FMC = 90^\circ$, and thus $CM$ is parallel to $FG$. Similarly, $CN$ is parallel to $FG$.

Thus, $MN$ is a straight line segment, and since it passes through $C$, the centre of the circle, $MN$ must also be a diameter of the circle. Thus, $MN = 1$. Also, $FMNG$ is a rectangle, so $FG = MN = 1$ and the perimeter of square $FGHJ$ is equal to $4 \times FG = 4(1) = 4$. This gives us an upper bound for $\pi$. That is, we know $\pi < 4$.

Therefore, a lower bound for $\pi$ is $4\sqrt{0.5} \approx 2.828$ and an upper bound for $\pi$ is 4. That is, $4\sqrt{0.5} < \pi < 4$.

**Note:** Since we know that $\pi \approx 3.14$, these are not the best bounds for $\pi$. Archimedes used regular polygons with more sides to get better bounds. In the Problem of the Week E problem, we investigate using regular hexagons to get better bounds.