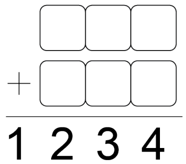


Problem of the Week

Problem D and Solution

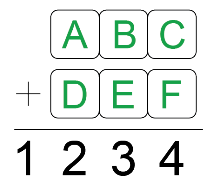
Arranging Tiles 2

Problem

Hugo has a box of tiles, each with an integer from 1 to 9 on it. Each integer appears on at least six tiles. Hugo creates larger numbers by placing tiles side by side. For example, using the tiles 3 and 7, Hugo can create the 2-digit number 37 or 73. Using six of his tiles, Hugo forms two 3-digit numbers that add to 1234. He then records the sum of the digits on the six tiles. How many different possible sums are there?

Solution

We will use the letters A, B, C, D, E, and F to represent the integers on the six chosen tiles, letting the two 3-digit numbers be ABC and DEF.



To solve this problem, we will look at each column starting with the units, then tens, and then finally the hundreds column.

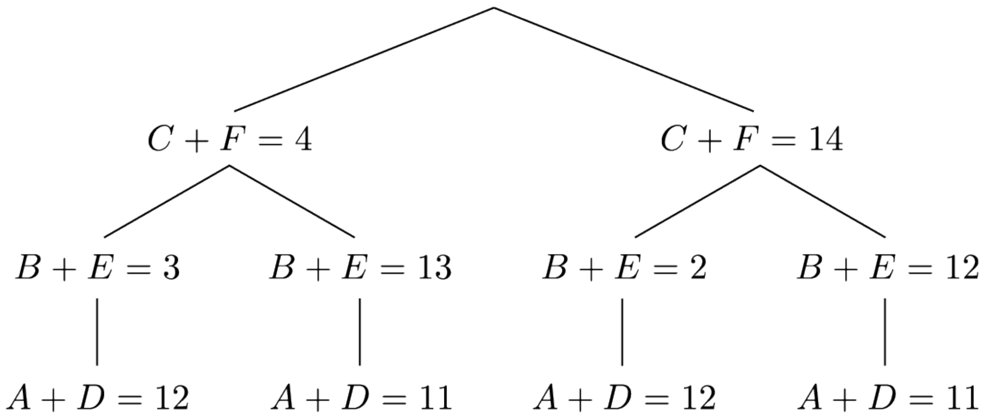
Since $C + F$ ends in a 4, then $C + F = 4$ or $C + F = 14$. The value of $C + F$ cannot be 20 or more, because C and F are digits. In the case that $C + F = 14$, we "carry" a 1 to the tens column. Now we will look at the tens column for these two cases.

- **Case 1:** $C + F = 4$
  Since the result in the tens column is 3 and there was no "carry" from the units column, it follows that $B + E$ ends in a 3. Then $B + E = 3$ or $B + E = 13$. The value of $B + E$ cannot be 20 or more, because B and E are digits. In the case that $B + E = 13$, we "carry" a 1 to the hundreds column.

- **Case 2:** $C + F = 14$
  Since the result in the tens column is 3 and there was a "carry" from the units column, it follows that $1 + B + E$ ends in a 3, so $B + E$ ends in a 2. Then $B + E = 2$ or $B + E = 12$. The value of $B + E$ cannot be 20 or more, because B and E are digits. In the case that $B + E = 12$, we "carry" a 1 to the hundreds column.

Since the result in the hundreds column is 12, then $A + D = 12$, or in the case when there was a "carry" from the tens column, $1 + A + D = 12$, so $A + D = 11$.

We summarize this information in the following tree.



Notice that if we add the three values along each of the four branches of the tree, we obtain the sum $(C + F) + (B + E) + (A + D)$, which is equal to $A + B + C + D + E + F$.

- The first branch has the sum $4 + 3 + 12 = 19$.
- The second branch has the sum $4 + 13 + 11 = 28$.
- The third branch has the sum $14 + 2 + 12 = 28$.
- The fourth branch has the sum $14 + 12 + 11 = 37$.

Therefore, there are 3 different values for the sum of the six digits. They are 19, 28, and 37.

Indeed, we can find values for the six digits that achieve each of these sums, as shown.

sum of 19	sum of 28	sum of 37
9 2 1	7 9 2	3 5 8
+ 3 1 3	+ 4 4 2	+ 8 7 6
1 2 3 4	1 2 3 4	1 2 3 4