Problem of the Week
Problem E and Solution
Overlapping Shapes 3

Problem
Austin draws $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm, and $\angle ABC = 90^\circ$. Lachlan then draws $\triangle DBF$ on top of $\triangle ABC$ so that $D$ lies on $AB$, $F$ lies on the extension of $BC$, $DB = 2$ cm, and sides $AC$ and $DF$ meet at $E$. If $AE = 3$ cm and $EC = 2$ cm, determine the length of $CF$.

Solution
Since $AB = 3$ and $DB = 2$, it follows that $AD = 1$ cm. Draw a perpendicular from $E$ to $BF$.
Let $P$ be the point where the perpendicular intersects $BF$. Let $CF = a$, $PC = b$, and $EP = h$.
We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles, the second uses trigonometry, and the third uses coordinate geometry.

Solution 1
Since $EP$ is perpendicular to $BF$, we know $\angle EPF = 90^\circ$. Also, $\angle ECP = \angle ACB$ (same angle). Therefore, $\triangle ABC \sim \triangle EPC$ (by angle-angle triangle similarity).

From the similarity, $\frac{AC}{BC} = \frac{EC}{PC}$, so $\frac{5}{4} = \frac{2}{b}$ or $b = \frac{8}{5}$. Also, $\frac{AC}{AB} = \frac{EC}{EP}$, so $\frac{5}{3} = \frac{2}{h}$ or $h = \frac{6}{5}$.

Now let’s calculate $PF$. We know $\angle EPF = \angle DBF = 90^\circ$ and $\angle EFP = \angle DFB$ (same angle).
Therefore, $\triangle DBF \sim \triangle EPF$ (by angle-angle triangle similarity). This tells us $\frac{DB}{BF} = \frac{EP}{PF}$.

Since $BF = BC + CF = 4 + a$ and $PF = PC + CF = \frac{8}{5} + a$, we have

$$\frac{DB}{BF} = \frac{EP}{PF}$$
$$\frac{2}{4 + a} = \frac{\frac{6}{5}}{\frac{8}{5} + a}$$
$$\frac{16}{5} + 2a = \frac{24}{5} + \frac{6a}{5}$$
$$2a - \frac{6a}{5} = \frac{24}{5} - \frac{16}{5}$$
$$\frac{4a}{5} = \frac{8}{5}$$
$$a = 2$$

Therefore, $CF = 2$ cm.
Solution 2

In $\triangle EPC$, $\sin(\angle ECP) = \frac{h}{2}$. In $\triangle ABC$, $\sin(\angle ACB) = \frac{3}{5}$.

Since $\angle ECP = \angle ACB$ (same angle),

$$\sin(\angle ECP) = \sin(\angle ACB)$$

$$\frac{h}{2} = \frac{3}{5}$$

$$h = \frac{6}{5}$$

Since $\triangle EPC$ is a right-angled triangle,

$$EP^2 + PC^2 = EC^2$$

$$h^2 + b^2 = 2^2$$

$$\left(\frac{6}{5}\right)^2 + b^2 = 4$$

$$b^2 = 4 - \frac{36}{25}$$

$$b^2 = \frac{64}{25}$$

$$b = \frac{8}{5}, \text{ since } b > 0$$

In $\triangle EPF$, $\tan(\angle EFP) = \frac{EP}{PF} = \frac{h}{a+b} = \frac{\frac{6}{5}}{a+\frac{8}{5}}$.

In $\triangle DBF$, $\tan(\angle DFB) = \frac{DB}{BF} = \frac{2}{4+a}$.

Since $\angle EFP = \angle DFB$ (same angle),

$$\tan(\angle EFP) = \tan(\angle DFB)$$

$$\frac{\frac{6}{5}}{a+\frac{8}{5}} = \frac{2}{4+a}$$

$$\frac{24}{5} + \frac{6}{5}a = 2a + \frac{16}{5}$$

$$2a - \frac{6}{5}a = \frac{24}{5} - \frac{16}{5}$$

$$\frac{4}{5}a = \frac{8}{5}$$

$$a = 2$$

Therefore, $CF = 2$ cm.
Solution 3

We will use coordinate geometry in this solution, and place $B$ at the origin. Using the given information, $D$ is at $(0, 2)$, $A$ is at $(0, 3)$, $C$ is at $(4, 0)$, and $F$ is on the positive $x$-axis at $(f, 0)$ with $f > 4$. Consider the circle through $E$ with centre $C(4, 0)$. Since $CE = 2$, the radius of this circle is 2. Thus, the equation of this circle is $(x - 4)^2 + y^2 = 4$.

The line passing through $A(0, 3)$ and $C(4, 0)$ has $y$-intercept 3 and slope $-\frac{3}{4}$, and so has equation $y = -\frac{3}{4}x + 3$. Since $E$ lies on the line with equation $y = -\frac{3}{4}x + 3$ and the circle with equation $(x - 4)^2 + y^2 = 4$, to find the coordinates of $E$, we substitute $y = -\frac{3}{4}x + 3$ for $y$ in $(x - 4)^2 + y^2 = 4$. Note that $E$ is in the first quadrant so $x > 0$ and $y > 0$.

Doing so, we get

$$(x - 4)^2 + \left(-\frac{3}{4}x + 3\right)^2 = 4$$

Expanding the left side, we get

$$x^2 - 8x + 16 + \frac{9}{16}x^2 - \frac{9}{2}x + 9 = 4$$

Multiplying by 16, we get

$$16x^2 - 128x + 256 + 9x^2 - 72x + 144 = 64$$

Simplifying, we get

$$25x^2 - 200x + 336 = 0$$

Factoring, we then get

$$(5x - 12)(5x - 28) = 0$$

It follows that $x = \frac{12}{5}$ or $x = \frac{28}{5}$. Substituting $x = \frac{12}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = \frac{6}{5}$.

Substituting $x = \frac{28}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = -\frac{6}{5}$. But $E$ is in the first quadrant so $y > 0$, and this second possibility is inadmissible. It follows that $E$ has coordinates $\left(\frac{12}{5}, \frac{6}{5}\right)$.

We can now find the equation of the line containing $D(0, 2)$, $E\left(\frac{12}{5}, \frac{6}{5}\right)$, and $F(f, 0)$. This line

has $y$-intercept 2, slope equal to $\frac{\frac{6}{5} - 2}{\frac{12}{5} - 0} = \frac{-\frac{4}{5}}{\frac{12}{5}} = -\frac{1}{3}$, and thus has equation $y = -\frac{1}{3}x + 2$.

The point $F(f, 0)$ lies on this line, so $0 = -\frac{1}{3}(f) + 2$, which leads to $f = 6$. Thus, the point $F$ has coordinates $(6, 0)$. Since $C$ is at $(4, 0)$ and $F$ is at $(6, 0)$, $CF = 2$. It turns out that $F$ also lies on the circle through $E$.

Therefore, $CF = 2$ cm.