Problem of the Week
Problem E and Solution
The Shortest Path

Problem
On the Cartesian plane, we draw grid lines at integer points along the $x$ and $y$ axes. We can then draw paths along these grid lines between any two points with integer coordinates. The graph below shows two paths along these grid lines from $O(0,0)$ to $P(6,-4)$. One path has length 10 and the other has length 20.

There are many different paths along the grid lines from $O$ to $P$, but the smallest possible length of such a path is 10. Let’s call this smallest possible length the path distance from $O$ to $P$.

Determine the number of points with integer coordinates for which the path distance from $O$ to that point is 10.

Solution
Solution 1
Let $Q(a,b)$ be a point that has path distance 10 from $O(0,0)$.

Let’s first suppose that $Q$ lies on the $x$ or $y$ axis.
The only point along the positive $x$-axis that has path distance 10 from the origin is $(10,0)$.
The only point along the negative $x$-axis that has path distance 10 from the origin is $(-10,0)$.
The only point along the positive $y$-axis that has path distance 10 from the origin is $(0,10)$.
The only point along the negative $y$-axis that has path distance 10 from the origin is $(0,-10)$.
Therefore, there are 4 points along the axes that have a path distance 10 from $O$.

Next, let’s suppose $a > 0$ and $b > 0$, so $Q$ is in the first quadrant.
Since the path distance from $O$ to $Q$ is 10, there must be a path from $O$ to $Q$ that moves a total of $r$ units to the right and $u$ units up (in some order) such that $r + u = 10$. This means that $Q$ is $r$ units to the right of $O$ and $u$ units up from $O$. In other words, $a = r$ and $b = u$, so $a + b = r + u = 10$. 

$\text{Diagram}$

$\text{Graph}$

$\text{Diagram}$
The points \((a, b)\) in the first quadrant that satisfy \(a + b = 10\) where \(a\) and \(b\) are integers are \((1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\). There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path distance 10 from \(O\).

By symmetry, there are 9 points in each quadrant that have path distance 10 from \(O\). In quadrant 2, the points are \((-1, 9), (-2, 8), (-3, 7), (-4, 6), (-5, 5), (-6, 4), (-7, 3), (-8, 2), (-9, 1)\). In quadrant 3, the points are \((-1, -9), (-2, -8), (-3, -7), (-4, -6), (-5, -5), (-6, -4), (-7, -3), (-8, -2), (-9, -1)\). In quadrant 4, the points are \((1, -9), (2, -8), (3, -7), (4, -6), (5, -5), (6, -4), (7, -3), (8, -2), (9, -1)\).

Therefore, there are a total of \(4 + (4 \times 9) = 40\) points with integer coordinates that have path distance 10 from \(O\).

Solution 2

We are permitted 10 moves to get from the origin to a point by travelling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We examine the cases based on the number of horizontal moves.

- **0 horizontal moves**: Since there are 0 horizontal moves, there are 10 vertical moves. There are two possible endpoints, \((0, 10)\) and \((0, -10)\).
- **1 horizontal move**: Since there is 1 horizontal move, there are 9 vertical moves. There are four possible endpoints, \((-1, 9), (-1, -9), (1, 9), and (1, -9)\).
- **2 horizontal moves**: Since there are 2 horizontal moves, there are 8 vertical moves. There are four possible endpoints, \((-2, 8), (-2, -8), (2, 8), and (2, -8)\).
- **3 horizontal moves**: Since there are 3 horizontal moves, there are 7 vertical moves. There are four possible endpoints, \((-3, 7), (-3, -7), (3, 7), and (3, -7)\).
- **4 horizontal moves**: Since there are 4 horizontal moves, there are 6 vertical moves. There are four possible endpoints, \((-4, 6), (-4, -6), (4, 6), and (4, -6)\).
- **5 horizontal moves**: Since there are 5 horizontal moves, there are 5 vertical moves. There are four possible endpoints, \((-5, 5), (-5, -5), (5, 5), and (5, -5)\).
- **6 horizontal moves**: Since there are 6 horizontal moves, there are 4 vertical moves. There are four possible endpoints, \((-6, 4), (-6, -4), (6, 4), and (6, -4)\).
- **7 horizontal moves**: Since there are 7 horizontal moves, there are 3 vertical moves. There are four possible endpoints, \((-7, 3), (-7, -3), (7, 3), and (7, -3)\).
- **8 horizontal moves**: Since there are 8 horizontal moves, there are 2 vertical moves. There are four possible endpoints, \((-8, 2), (-8, -2), (8, 2), and (8, -2)\).
- **9 horizontal moves**: Since there are 9 horizontal moves, there is 1 vertical move. There are four possible endpoints, \((-9, 1), (-9, -1), (9, 1), and (9, -1)\).
- **10 horizontal moves**: Since there are 10 horizontal moves, there are 0 vertical moves. There are two possible endpoints, \((-10, 0)\) and \((10, 0)\).

Therefore, there are a total of \(2 + (4 \times 9) = 40\) points with integer coordinates that have path distance 10 from \(O\).