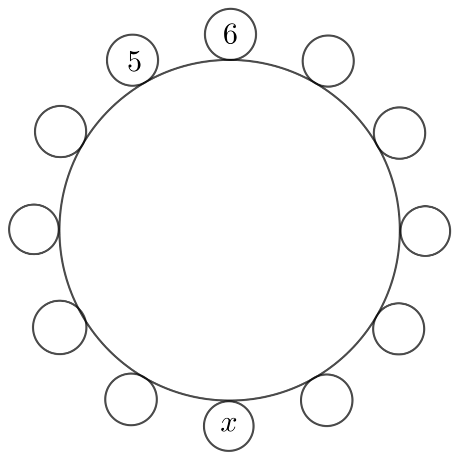
Problem of the Week
Problem E and Solution
Take a Seat 3

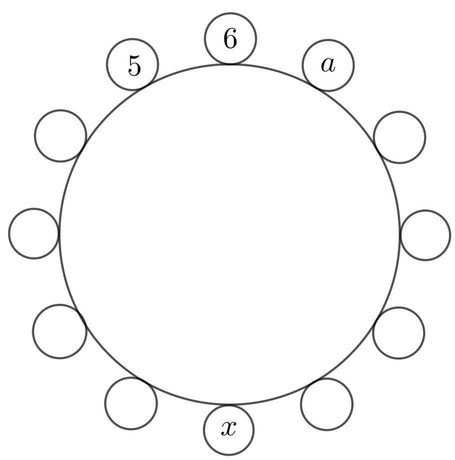
Problem

Twelve people are sitting, equally spaced, around a circular table. They each hold a card with a different integer on it. For any two people sitting beside each other, the positive difference between the integers on their cards is no more than 2. The people holding the integers 5 and 6 are seated as shown. The person opposite the person holding the 6 is holding the integer x. What are the possible values of x?



Solution

We will start with the card with the integer 6. We are given that 5 is on one side of the 6. Let a be the integer on the other side of the 6.



Since each card contains a different integer and the positive difference between the integers on two cards beside each other is no more than 2, then a must be 4, 7, or 8. We will consider these three cases.

Case 1: $a = 7$

Since the number on each card is different and we know that someone is holding a card with a 5 and someone is holding a card with a 6, then the integer to the right of the 7 must be 8 or 9.

Furthermore, every integer to the right of 7 must be greater than 7. Similarly, the integer to left of 5 is either 3 or 4. Furthermore, any integer to the left of 5 must be less than 5.

Since x is both to the right of 7 and to the left of 5, it must be both greater than 7 and less than 5. This is not possible.

Therefore, when $a = 7$, there is no solution for x.

Case 2: $a = 4$

Since the number on each card is different and we know that someone is holding a card with a 5 and someone is holding a card with a 6, then the integer to the right of 4 must be 3 or 2. We will look at these two subcases.

- **Case 2a:** The card with integer 3 is to the right of the card with integer 4.

  Notice then that every integer to the right of the 3 must be less than 3. Also the integer to the left of 5 must be 7 and every integer to left of the 7 must be greater than 7. Since x is both to the right of 3 and to the left of 7, it must be both greater than 7 and less than 3. This is not possible.

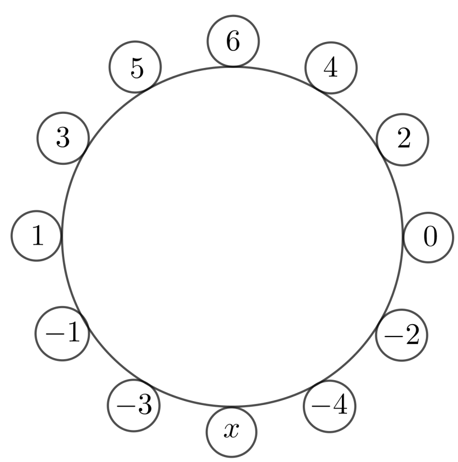
  Therefore, when $a = 4$ and the integer to the right of it is 3, there is no solution for x.

- **Case 2b:** The card with integer 2 is to the right of the card with integer 4.

  Now, the integer to the left of 5 can be either 7 or 3.

  If the integer is 7, then using a similar argument to that in Case 2a, there is no solution for x.

  If the integer to the left of 5 is 3, the only possible integer to the left of 3 is 1. This means the only possible integer to the right of 2 is 0. Which leads to the only possible integer to the left of 1 is -1. Furthermore, the only possible integer to the right of 0 is -2. Continuing in this manner, we get the table set up shown below.

  From here, the only possible solution is $x = -5$.

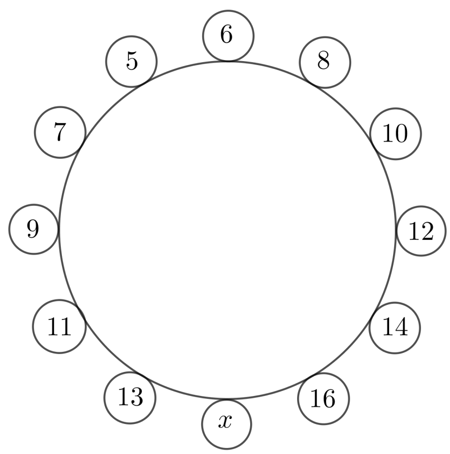
Therefore, when $a = 4$, the solution is $x = -5$.

Case 3: $a = 8$

Since the number on each card is different and we know that someone is holding a card with a 6, then the integer to right of the 8 must be 7, 9, or 10. Furthermore, since someone is already holding a 5 and someone is already holding a 6, every other integer to the the right of 8 must be 7 or greater.

The integer to the left of 5 is either 3, 4, or 7. If it is 3 or 4, then since someone is already holding the 5 and someone is already holding the 6, every integer to the left of 5 must be less than 5. Since x is both to the right of 8 and to the left of 5, if there is a 3 or a 4 to the left of 5, then x must be both 7 or greater and less than 5. This is not possible.

Therefore, if a solution exists when $a = 8$, then the integer to the left of 5 must be 7. The integer to the left of 7 must be 5, 6, 8, or 9. Since the 5, 6, and 8 are already placed, then the only possible integer to the left of 7 is 9. Similarly, the only possible integer to the right of 8 is 10. Thus, the integer to the left of 9 must be 11. Continuing in this manner, we get the table set up shown below.



From here, the only possible solution is $x = 15$.

Therefore, when $a = 8$, the solution is $x = 15$.

Therefore, the possible values for x are $x = -5$ or $x = 15$.