Problem of the Week
Problem E and Solution
Wipe Away 3

Problem
Tyra writes consecutive positive integers on a whiteboard starting with the integer 1. However, when she writes a number that is a multiple of 9, or contains the digit 9, Juliana immediately erases it. If they continue this for a long time, what is the 400th number that Juliana will erase?

Note: In solving this problem, it may be helpful to use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9. For example, the number 214578 is divisible by 9 since \(2 + 1 + 4 + 5 + 7 + 8 = 27\), which is divisible by 9. In fact, \(214578 = 9 \times 23842\).

Solution
We first consider the integers between 1 and 999, inclusive. Since 999 = 111 \times 9\), there are 111 multiples of 9 between 1 and 999.

Now let’s figure out how many of the integers from 1 to 999 contain the digit 9. The integers from 1 to 99 that contain the digit 9 are 9, 19, \ldots, 79, 89 as well as 90, 91, \ldots, 97, 98, 99. Thus, there are 19 positive integers from 1 to 99 that contain the digit 9. Since there are 19 integers from 1 to 99 that contain the digit 9, it follows that there are 19 \times 9 = 171 integers from 1 to 899 that contain the digit 9.

Between 900 and 999, every integer contains the digit 9. Thus, there are 100 numbers that contain the digit 9. Thus, in total, \(171 + 100 = 271\) of the integers from 1 to 999 contain the digit 9.

However, some of the integers that contain the digit 9 are also multiples of 9, so were counted twice. To determine how many of these such numbers there are, we use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9.

- The only one-digit number that contains the digit 9 and is also a multiple of 9 is 9 itself.
- The only two-digit numbers that contain the digit 9 and are also multiples of 9 are 90 and 99.
- To find the three-digit numbers that contain the digit 9 and are also multiples of 9, we will look at their digit sum.

  - Case 1: Three digit-numbers with a digit sum of 9:
    The only possibility is 900. Thus, there is 1 number.
  - Case 2: Three digit-numbers with a digit sum of 18:
    * If two of the digits are 9, then the other digit must be 0. The only possibilities are 909 and 990. Thus, there are 2 numbers.
    * If only one of the digits is 9, then the other two digits must add to 9. The possible digits are 9, 4, 5, or 9, 3, 6, or 9, 2, 7, or 9, 8, 1. For each of these sets of digits, there are 3 choices for the hundreds digit. Once the hundreds
digit is chosen, there are 2 choices for the tens digit, and then the remaining
digit must be the ones digit. Thus, there are $3 \times 2 = 6$ possible three-digit
numbers for each set of digits. Since there are 4 sets of digits, then there are
$4 \times 6 = 24$ possible numbers.

- **Case 3:** Three digit-numbers with a digit sum of 27:
The only possibility is 999. Thus, there is 1 number.

Therefore, there are $1 + 2 + 24 + 1 = 28$ three-digit numbers from 1 to 999 that contain
the digit 9, and are also multiples of 9.

Thus, there are $1 + 2 + 28 = 31$ integers from 1 to 999 that contain the digit 9, and are also
multiples of 9. It follows that Juliana erases $111 + 271 - 31 = 351$ of the numbers from 1 to
999 from the whiteboard. Since we are looking for the 400th number that Juliana erases, we
need to keep going.

Next, we consider the integers between 1000 and 1099, inclusive. Since $1099 = (122 \times 9) + 1$,
there are 122 multiples of 9 between 1 and 1099. Since there are 111 multiples of 9 between 1
and 999, it follows that there are $122 - 111 = 11$ multiples of 9 between 1000 and 1099. The
integers between 1000 and 1099 that contain the digit 9 are 1009, 1019, . . . , 1079, 1089 as well
as 1090, 1091, . . . , 1097, 1098, 1099. Thus, there are 19 integers from 1000 to 1099 that
contain the digit 9. Of these, the only integers that are also multiples of 9 are 1089 and 1098.
Thus, Juliana erases $11 + 19 - 2 = 28$ of the numbers from 1000 to 1099 from the whiteboard.
In total, she has now erased $351 + 28 = 379$ numbers.

Next, we consider the integers between 1100 and 1189, inclusive. Since $1189 = (132 \times 9) + 1$,
there are 132 multiples of 9 between 1 and 1099. Since there are 122 multiples of 9 between 1
and 1099, it follows that there are $132 - 122 = 10$ multiples of 9 between 1100 and 1189. The
integers between 1100 and 1189 that contain the digit 9 are 1109, 1119, . . . , 1179, 1189. Thus
there are 9 integers from 1100 to 1189 that contain the digit 9. The only one of these that is
also a multiple of 9 is 1179. Thus, Juliana erases $10 + 9 - 1 = 18$ of the numbers from 1100 to
1189 from the whiteboard. In total, she has now erased $379 + 18 = 397$ numbers.

The next three numbers that Juliana will erase are 1190, 1191, and 1192. Thus, the 400th
number that Juliana erases is 1192.